sjst.scst.edu.ly
Surman **J**ournal for **S**cience and **T**echnology
ISSN: Online (2790-5721) - Print (2790-5713)
مجلة صرمان للعلوم والتقنية
Vol**6**, No.**2**, Jun. – Dec. 2024
Pages: 401 ~ 410

Some Applications of Catas Operator to P-valent Starlike Functions

Abdusalam R. Ahmed
Department of Mathematics- Faculty of Science - Azzaytuna University
Adrm1980@yahoo.com

الملخص
في هذا البحث نقدم دراسة على فصول جزئية جديدة للدوال النجمية والمحدبة وشبه وقريبة التحدب متعددة القيمة معرفة بواسطة مؤثر كاتاس في قرص الوحدة $U = \{z \in ¢: |z| < 1\}$.
تم استنتاج بعض الخواص والحصول على نتائج علاقات التضمين لفصول تلك الدوال متضمنة مؤتر كاتاس. نتائج تأثير المؤثر التكاملي للدوال المركبة في تلك الفصول الجزئية ايضاً تم مناقشتها واثباتها.
ايضاً تم تعريف تركيب المؤثرات المركبة مؤثر كاتاس مع مؤثر ليبرا التكاملي وتعريف المؤثر المركب على الفصول الجزئية للدوال المركبة النجمية والمحدبة وشبه وقريبة التحدب متعددة القيمة في قرص الوحدة U ، وتم الحصول على متطابقات تلك المؤثرات واستخدامها في استنتاج بعض الخواص المتعلقة.
كذلك تم استخدام بعض النظريات المساعدة مع متطابقات المؤثرات المعرفة في الحصول على بعض الخصائص واثبات النتائج الأساسية المتعلقة بفصول تلك الدوال.

Abstract.
The purpose of the present paper, is to investigate and study some new subclasses of p-valent starlike, convex, close-to-convex, and quasi-convex functions associated with Catas operator in the open unit disk $U = \{z \in ¢: |z| < 1\}$.
Inclusion relations are established, defined the integral operator of functions in these subclasses and some properties of them are discussed.
Also defined combining operation between Catas operator with Libera integral operator for these subclasses in U, also we use some Lammas and identities of operators to get and proof main results of these subclasses.

***Key Words**: Analytic Functions – Starlike, Convex, Close – to – Convex Functions – Catas Operator.*

1. Introduction:

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N = \{1,2,3, \dots\}) \qquad (1.1)$$

which are analytic and p-valent in the open unit disc $U = \{z: z \in \mathbb{C} \ and \ |z| < 1\}$.
Also the Hadamard product or (convolution) of two functions

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p} \qquad (j = 1,2),$$

given by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,1}\, a_{n+p,2} z^{n+p} = (f_2 * f_1)(z). \qquad (1.2)$$

A function $f(z) \in A(p)$ is called p-valent starlike of order α if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \le \alpha < p, z \in U) \qquad (1.3)$$

We denote by $S_p^*(\alpha)$ the class of all p-valent starlike functions of order α.
We note that $S_p^*(0) = S_p^*$ the class of p-valent starlike functions in U.
A function $f(z) \in A(p)$ is called p-valent convex function of order α if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \le \alpha < p, z \in U) \qquad (1.4)$$

We denote by $C_p(\alpha)$ the class of all p-valent convex functions of order α. We note that $C_p(0) = C_p$, the class of p-valent convex functions in U.

It follows from (1.3) and (1.4) that

$$f(z) \in C_p(\alpha) \; iff \; \frac{zf'(z)}{p} \in S_p^*(\alpha) \qquad (0 \leq \alpha < p) \tag{1.5}$$

The classes S_p^* and C_p were introduced by Deniz [2]. Furthermore, a function $f(z) \in A(p)$ is said to be p-valent close-to-convex function of order β and type γ if there exists a function $g(z) \in S_p^*(\gamma)$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \beta \qquad (0 \leq \beta, \gamma < p, z \in U). \tag{1.6}$$

We denote this class by $k_p(\beta,\gamma)$. The class $k_p(\beta,\gamma)$ was studied by Aouf [3].

A function $f(z) \in A(p)$ is called quasi-convex of order β type γ, if there exist a function $g(z) \in C_p(\gamma)$ such that

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta, \quad (0 \leq \beta, \gamma < p, z \in U) \tag{1.7}$$

We denote this class $k_p^*(\beta,\gamma)$. Clearly

$$f(z) \in k_p^*(\beta,\gamma) \Leftrightarrow \frac{zf'(z)}{p} \in k_p(\beta,\gamma)$$

let $f(z) \in A(p)$. For $p \in N$, δ, λ and $\ell \geq 0$. Catas [1], defined the multiplier transformations $I_P(\delta,\lambda,\ell)$ on $A(p)$ by the following infinite series:

$$I_p(\delta,\lambda,\ell)f(z) \coloneqq z^p + \sum_{n=1}^{\infty}\left[\frac{p+\lambda n+\ell}{p+\ell}\right]^{\delta} a_{n+p}z^{n+p}. \tag{1.8}$$

It is easily verified from (1.8)

$$z\left(I_p(\delta,\lambda,\ell)f(z)\right)' = \left(\frac{p+\ell}{\lambda}\right)I_p(\delta+1,\lambda,\ell)f(z) - \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]I_p(\delta,\lambda,\ell)f(z) \tag{1.9}$$

Also if $f(z)$ is given by (1.1), then we have

$$I_p(\delta,\lambda,\ell)f(z) = \left(f * \varphi_{p,\lambda,\ell}^{\delta}\right)(z), \tag{1.10}$$

where

$$\varphi_{p,\lambda,\ell}^{\delta}(z) = z^p + \sum_{n=1}^{\infty}\left[\frac{p+\lambda n+\ell}{p+\ell}\right]^{\delta} z^{n+p} \tag{1.11}$$

Using the operator $I_p(\delta,\lambda,\ell)f(z)$ is given by (1.10), we introduce the following subclasses of p-valent function:

$$\begin{cases} S_p^*(\gamma,\delta,\lambda,\ell) = \{f(z) \in A(p): I_p(\delta,\lambda,\ell)f(z) \in S_p^*(\gamma)\} \\ C_p(\gamma,\delta,\lambda,\ell) = \{f(z) \in A(p): I_p(\delta,\lambda,\ell)f(z) \in C_p(\gamma)\} \\ k_p(\beta,\gamma,\delta,\lambda,\ell) = \{f(z) \in A(p): I_p(\delta,\lambda,\ell)f(z) \in k_p(\beta,\gamma)\} \\ k_p^*(\beta,\gamma,\delta,\lambda,\ell) = \{f(z) \in A(p): I_p(\delta,\lambda,\ell)f(z) \in k_p^*(\beta,\gamma)\} \end{cases}$$

2. Inclusion Relation

We need some lemmas to prove our results.

Lemma 2.1 [4]. Let $w(z)$ be regular in the unit disk U, with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r$ at a point $z_0 \in U$, we can write $z_0 w'(z_0) = k w(z_0)$, where k is real and $k \geq 1$.

Lemma 2.2 [5]. Let $\varphi(u,v)$ be a complex function, $\varphi: D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$, and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that φ satisfies the following conditions:

i. $\varphi(u,v)$ is continuous in D.
ii. $(1,0) \in D$ and $Re[\varphi(1,0)] > 0$.
iii. $Re\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+{u_2}^2)$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in U, such that $(h(z), zh(z)) \in D$ for all $z \in U$. If $Re\{\varphi(h(z), zh'(z))\} > 0$, $(z \in U)$ then $Re\, h(z) > 0$ for $z \in U$.

Theorem 2.3. $S_p^*(\gamma, \delta+1, \lambda, \ell) \subset S_p^*(\gamma, \delta, \lambda, \ell)$ for any complex number δ.

Proof:

Let $f(z) \in S_p^*(\gamma, \delta+1, \lambda, \ell)$, and set

$$\frac{z(I_p(\delta,\lambda,\ell)f(z))'}{I_p(\delta,\lambda,\ell)f(z)} = \gamma + (p-\gamma)h(z), 0 \leq \gamma < p, Z \in U \tag{2.1}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $h(0)=1, h(z) \neq 0$, for all $z \in U$. From (1.9), we can write

$$\begin{cases} \dfrac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)} = \dfrac{\lambda}{p+\ell}\left[\dfrac{z\left(I_p(\delta,\lambda,\ell)f(z)\right)'}{I_p(\delta,\lambda,\ell)f(z)} + \dfrac{p(1-\lambda)+\ell}{\lambda}\right] \\ \dfrac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)} = \dfrac{\lambda}{p+\ell}\left[\gamma + (p-\gamma)h(z) + \dfrac{p(1-\lambda)+\ell}{\lambda}\right] \end{cases} \tag{2.2}$$

By logarithmically differentiating both sides of the equation (2.2) and multiplying by z, we have

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)f(z)} - \gamma = (p-\gamma)h(z) + \frac{(p-\gamma)zh'(z)}{\left[(p-\gamma)h(z)+\gamma-p+\frac{1}{\lambda}(p+\ell)\right]} \tag{2.3}$$

Taking $h(z) = u = u_1 + iu_2$ and $zh'(z) = v = v_1 + iv_2$ we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = (p-\gamma)u + \frac{(p-\gamma)v}{(p-\gamma)u + \left(\frac{p+\ell}{\lambda} - p\right) + \gamma} \tag{2.4}$$

this implies

i. $\varphi(u,v)$ is continuous in $D = \left(\mathbb{C} - \frac{\left(\frac{p+\ell}{\lambda}-p\right)+\gamma}{\gamma - p}\right) \times \mathbb{C}$.

ii. $(1,0) \in D$ and $Re\{\varphi(1,0)\} > 0$.

To verity the condition (iii), we calculate as follows:

$$Re\{\varphi(iu_2, v_1)\} = Re\left\{\frac{(p-\gamma)v_1}{(p-\gamma)iu_2 + \left(\frac{p+\ell}{\lambda} - p\right) + \gamma}\right\}$$

$$= Re\left\{\frac{(p-\gamma)v_1\left[\left(\frac{p+\ell}{\lambda} - p + \gamma\right) - i(p-\gamma)u_2\right]}{(p-\gamma)^2 u_2^2 + \left(\frac{p+\ell}{\lambda} - p + \gamma\right)^2}\right\}$$

$$= Re\left\{\frac{(p-\gamma)\left(\frac{p+\ell}{\lambda}-p+\gamma\right)v_1-i(p-\gamma)^2v_1u_2}{(p-\gamma)^2u_2^2+\left(\frac{p+\ell}{\lambda}-p+\gamma\right)^2}\right\} \tag{2.5}$$

$$=\left\{\frac{(p-\gamma)\left(\frac{p+\ell}{\lambda}-p+\gamma\right)v_1}{(p-\gamma)^2u_2^2+\left(\frac{p+\ell}{\lambda}-p+\gamma\right)^2}\right\}$$

$$\le \frac{-(p-\gamma)\left(\frac{p+\ell}{\lambda}-p+\gamma\right)(1+u_2^2)}{2\left[(p-\gamma)^2u_2^2+\left(\frac{p+\ell}{\lambda}-p+\gamma\right)^2\right]}<0,$$

for all $(iu_2, v_1) \in D$ such that $v_1 \le \frac{-1}{2}(1+u_2^2)$. Hence, the function $\varphi(u,v)$ satisfies the conditions of Lemma 2.2. This shows that if $Re\{\varphi(h(z), zh'(z)\} > 0, z \in U$, then

$$Re\big(h(z)\big) > 0,\ (z \in U) \tag{2.6}$$

$$Re\left(\frac{z\big(I_p(\delta,\lambda,\ell)f(z)\big)'}{I_p(\delta,\lambda,\ell)f(z)}\right) = Re(\gamma) + Re(p-\gamma)R\, h(z)$$

$$Re\left(\frac{z\big(I_p(\delta,\lambda,\ell)f(z)\big)'}{I_p(\delta,\lambda,\ell)f(z)}\right) > \gamma$$

So, $f(z) \in S_p^*(\gamma,\delta,\lambda,\ell)$

hence $S_p^*(\gamma,\delta+1,\lambda,\ell)f(z) \subset S_p^*(\gamma,\delta,\lambda,\ell)f(z)$ ▪ (2.7)

Theorem 2.4. $C_p(\gamma,\delta+1,\lambda,\ell) \subset C_p(\gamma,\delta,\lambda,\ell)$, for any complex number δ.

Proof:

Consider the following

$$\begin{aligned}
f(z) \in C_p(\gamma,\delta+1,\lambda,\ell) &\Leftrightarrow I_p(\delta+1,\lambda,\ell)f(z) \in C_p(\gamma)\\
&\Leftrightarrow \frac{z}{p}\Big(I_p(\delta+1,\lambda,\ell)f(z)\Big)' \in S_p^*(\gamma)\\
&\Leftrightarrow I_p(\delta+1,\lambda,\ell)\left(\frac{zf'(z)}{p}\right) \in S_p^*(\gamma)\\
&\Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\gamma,\delta+1,\lambda,\ell)\\
&\Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\gamma,\delta,\lambda,\ell)\\
&\Leftrightarrow I_p(\delta,\lambda,\ell)\left(\frac{zf'(z)}{p}\right) \in S_p^*(\gamma) \qquad (2.8)\\
&\Leftrightarrow \frac{z}{p}(I(\delta,\lambda,\ell)f(z))' \in S_p^*(\gamma)\\
&\Leftrightarrow I(\delta,\lambda,\ell)f(z) \in C_p(\gamma)\\
&\Leftrightarrow f(z) \in C_p(\gamma,\delta,\lambda,\ell)
\end{aligned}$$

The proof is completed.

Theorem 2.5 $k_p(\beta,\gamma,\delta+1,\lambda,\ell) \subset k_p(\beta,\gamma,\delta,\lambda,\ell)$, for any complex number δ.

Proof:

Let $f(z) \in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$. Then, there exists a function $k(z) \in S_p^*(\gamma)$, such that

$$Re\left\{\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{g(z)}\right\} > \beta \qquad (z \in U) \tag{2.9}$$

$I_p(\delta+1,\lambda,\ell)k(z) = g(z)$ and

$$Re\left\{\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)k(z)}\right\} > \beta \quad (z \in U).$$

Now, put

$$\frac{z\left(I_p(\delta,\lambda,\ell)f(z)\right)'}{I_p(\delta,\lambda,\ell)k(z)} = \beta + (p-\beta)h(z), \qquad 0 \le \beta < 1, z \in U$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ using the identity (1.9) we have

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)k(z)} = \frac{I_p(\delta+1,\lambda,\ell)(zf'(z))}{I_p(\delta+1,\lambda,\ell)k(z)}$$

$$= \frac{z\left[I_p(\delta,\lambda,\ell)(zf'(z))\right]' + \frac{1}{\lambda}[p(1-\lambda)+\ell]I_p(\delta,\lambda,\ell)(zf'(z))}{z\left[I_p(\delta,\lambda,\ell)k(z)\right]' + \frac{1}{\lambda}[p(1-\lambda)+\ell]I_p(\delta,\lambda,\ell)k(z)}$$

$$= \frac{\dfrac{z\left[I_p(\delta,\lambda,\ell)(zf'(z))\right]'}{I_p(\delta,\lambda,\ell)k(z)} + \dfrac{[p(1-\lambda)+\ell]I_p(\delta,\lambda,\ell)(zf'(z))}{\lambda I_p(\delta,\lambda,\ell)k(z)}}{\dfrac{z\left[I_p(\delta,\lambda,\ell)k(z)\right]'}{I_p(\delta,\lambda,\ell)k(z)} + \dfrac{1}{\lambda}[p(1-\lambda)+\ell]}, \tag{2.10}$$

since $k(z) \in S_p^*(\gamma,\delta+1,\lambda,\ell) \subset S_p^*(\gamma,\delta,\lambda,\ell)$, we let

$$\frac{z\left(I_p(\delta,\lambda,\ell)k(z)\right)'}{I_p(\delta,\lambda,\ell)k(z)} = \gamma + (p-\gamma)H(z),$$

where $Re\, H(z) > 0,\ (z \in U)$ thus can be written as

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)k(z)} = \frac{\dfrac{z\left(I_p(\delta,\lambda,\ell)(zf'(z))\right)'}{I_p(\delta,\lambda,\ell)k(z)} + \left[\dfrac{p+\ell}{\lambda} - p\right][\beta + (p-\beta)h(z)]}{[\gamma + (p-\gamma)H(z)] + \left[\dfrac{p+\ell}{\lambda} - p\right]}, \tag{2.11}$$

consider that

$$z\left(I_p(\delta,\lambda,\ell)f(z)\right)' = I_p(\delta,\lambda,\ell)k(z)[\beta + (p-\beta)h(z)]. \tag{2.12}$$

Differentiating (2.12), and multiplying by z, we have

$$\frac{z\left(I_p(\delta,\lambda,\ell)(zf')(z)\right)'}{I_p(\delta,\lambda,\ell)k(z)} = (p-\beta)zh'(z) + [\beta + (p-\beta)h(z)][\gamma + (p-\gamma)H(Z)], \tag{2.13}$$

using (2.13) and (2.11), we have

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)k(z)} = [\beta + (p-\beta)h(z)] + \frac{(p-\beta)zh'(z)}{[\gamma + (p-\gamma)H(Z)] + \left(\frac{p+\ell}{\lambda} - p\right)} \tag{2.14}$$

Taking $h(z) = u$ and $zh'(z) = v$ in (2.14), we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = (p-\beta)u + \frac{(p-\beta)v}{[\gamma+(p-\gamma)H(z)]+\left(\frac{p+\ell}{\lambda}-p\right)}, \tag{2.15}$$

this implies

i. $\varphi(u,v)$ is continuous in $D = \mathbb{C}\times\mathbb{C}$.

ii. $(1,0) \in D$ and $Re\{\varphi(1,0)\} > 0$.

To verify condition (iii), we proceed as follows:

$$\varphi(iu_2, v_1) = \frac{(p-\beta)v_1\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y) - i(p-\gamma)h_2(x,y)\right]}{\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y)\right]^2 + [(p-\gamma)h_2(x,y)]^2}$$

$$Re\,\varphi(iu_2, v_1) = \frac{(p-\beta)v_1\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y)\right]}{\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y)\right]^2 + [(p-\gamma)h_2(x,y)]^2} \tag{2.16}$$

Where $H(z) = h_1(x,y) + ih_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being the functions of x and y and $Re\,H(z) = h_1(x,y) > 0$. Since $v_1 \le \frac{-1}{2}(1+u_2^2)$, impties

$$Re\left(\varphi(iu_2, v_1)\right) = \frac{-(p-\beta)(1+u_2^2)\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y)\right]}{2\left[\left[\gamma + \left(\frac{p+\ell}{\lambda} - p\right) + (p-\gamma)h_1(x,y)\right]^2 + [(p-\gamma)h_2(x,y)]^2\right]} < 0 \tag{2.17}$$

Hence, $Reh(z) > 0$ $(z \in U)$ and $f(z) \in k_p(\beta,\gamma,\delta,\lambda,\ell)$▪

Theorem 2.6 $k_p^*(\beta,\gamma,\delta+1,\lambda,\ell) \subset k_p^*(\beta,\gamma,\delta,\lambda,\ell)$ for any complex number δ.

Proof:

Consider the following:

$$\begin{aligned} f(z) \in k_p^*(\beta,\gamma,\delta+1,\lambda,\ell) &\Leftrightarrow I_p(\delta+1,\lambda,\ell)f(z) \in k_p^*(\beta,\gamma) \\ &\Leftrightarrow \frac{z}{p}\left(I_p(\delta+1,\lambda,\ell)f(z)\right)' \in k_p(\beta,\gamma) \\ &\Leftrightarrow I_p(\delta+1,\lambda,\ell)\left(zf'(z)\right) \in k_p(\beta,\gamma) \\ &\Rightarrow \frac{zf'(z)}{p} \in k_p(\beta,\gamma,\delta+1,\lambda,\ell) \end{aligned}$$

SJST
sjst.scst.edu.ly
Surman Journal for Science and Technology
ISSN: Online (2790-5721) - Print (2790-5713)
مجلة صرمان للعلوم والتقنية
Vol6, No.2, Jun. – Dec. 2024
Pages: 401 ~ 410

$$\Rightarrow \frac{zf\prime(z)}{p}) \in k_p(\beta,\gamma,\delta,\lambda,\ell) \qquad (2.18)$$

$$\Leftrightarrow I_p(\delta,\lambda,\ell)\left(\frac{zf'(z)}{p}\right) \in k_p(\beta,\gamma)$$

$$\Leftrightarrow \frac{z}{p}\left(I_p(\delta,\lambda,\ell)f(z)\right)' \in k_p(\beta,\gamma)$$

$$\Leftrightarrow I_p(\delta,\lambda,\ell)f(z) \in k_p^*(\beta,\gamma)$$

$$\Rightarrow f(z) \in k_p^*(\beta,\gamma,\delta,\lambda,\ell) \ ▪$$

3. Integral operator

For $c > 1$ and $(z) \in A(p)$, we recall here the generalized Bernardi-Libera- Livingston integral operator $L_c f(z)$ as follows

$$L_c f(z) = \frac{c+p^z}{z^c} t^{c-1} f(t)dt$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right) a_{n+p} z^{n+p} \qquad (3.1)$$

The operator $L_c(f(z))$ when $c \in N = \{1,2,3, \ldots\}$ was studied by Bernardi [6], for $c = 1$, $L_1(f(z))$ was investigated earlier by Libera [7]. Now, we have

$$I_p(\delta,\lambda,\ell)(L_c f(z)) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+\lambda n+\ell}{p+\ell}\right)\left(\frac{c+p}{c+p+n}\right) a_{n+p} z^{n+p} \qquad (3.2)$$

so, we get the identity

$$\left(I_p(\delta,\lambda,\ell)\left(L_c f(z)\right)\right)' = \left(\frac{p+\ell}{\lambda}\right) I_p(\delta+1,\lambda,\ell)f(z) - \left(\frac{p+\ell}{\lambda} - p\right) I_p(\delta,\lambda,\ell)L_c f(z) \qquad (3.3)$$

The following theorems deal with the generalized Bernardi-Libera- Livingston integral operator $L_c(f(z))$ defined by (3.1).

Theorem 3.1

Let $c > -\gamma, 0 \le \gamma < p$. If $f(z) \in S_p^*(\gamma,\delta+1,\lambda,\ell)$, then $L_c f(z) \in S_p^*(\gamma,\delta+1,\lambda,\ell)$

Proof:

From (3.3), we have

$$\frac{z\left(I_p(\delta,\lambda,\ell)L_c f(z)\right)'}{I_p(\delta,\lambda,\ell)L_c f(z)} = \frac{(p+\ell)I_p(\delta+1,\lambda,\ell)f(z)}{\lambda\, I_p(\delta,\lambda,\ell)L_c f(z)} - \left(\frac{p+\ell}{\lambda} - p\right)$$

$$= \frac{1+(1-2\gamma)w(z)}{1-w(z)} \qquad (3.4)$$

Then $w(z)$ is analytic in U, $w(0) = 0$. Using (3.3) and (3.4) we get

$$\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)L_c f(z)} = \frac{(\lambda+p+\ell-p\lambda)+(\lambda-2\lambda\gamma-p-\ell+p\lambda)w(z)}{(p+\ell)(1-w(z))} \qquad (3.5)$$

Differentiating (3.5), we get

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta,\lambda,\ell)L_c f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} + \frac{zw\prime(z)}{1-w(z)} + \frac{(\lambda-2\lambda\gamma-p-\ell+p\lambda)zw\prime(z)}{(\lambda+p+\ell-p\lambda)+(\lambda-2\lambda\gamma-p-\ell+p\lambda)w(z)} \qquad (3.6)$$

Suppose that for $z_0 \in U$, $\max|w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0), k \ge 1$. Putting $z = z_0$ and $w(z_0) = e^{i\theta}$ in (3.6), we obtain

$$Re\left\{\frac{z_0\left(I_p(\delta+1,\lambda,l)f(z)\right)'}{I_p(\delta+1,\lambda,l)f(z)}-\gamma\right\}=Re\left\{\frac{(1-\gamma)(1+e^{i\theta})+ke^{i\theta}}{1-e^{i\theta}}\right\}+Re\left\{\frac{(\lambda-2\lambda\gamma-p-\ell+p\lambda)ke^{i\theta}}{(\lambda+p+\ell-p\lambda)+(\lambda-2\lambda\gamma-p-\ell+p\lambda)e^{i\theta}}\right\} \quad (3.7)$$

$$=\frac{-k[(\lambda+p+\ell-p\lambda)^2-(\lambda-2\lambda\gamma-p-\ell+p\lambda)^2]}{2(\lambda+p+\ell-p\lambda)^2+4\ell(\lambda-2\lambda\gamma-p-\ell+p\lambda)\cos\theta+2(\lambda-2\lambda\gamma-p-\ell+p\lambda)^2}\leq 0$$

Which contradicts the hypothesis that $f(z)\in S_p^*(\gamma,\delta+1,\lambda,\ell)$. Hence, $|w(z)|<1$, for $z\in U$, and it follows (3.4), that $L_cf(z)\in S_p^*(\gamma,\delta+1,\lambda,\ell)$.

Theorem 3.2

Let $c>-\gamma, 0\leq\gamma<p$. If $f(z)\in C_p(\gamma,\delta+1,\lambda,\ell)$, then $L_cf(z)\in C_p(\gamma,\delta+1,\lambda,\ell)$

Proof:

$$\begin{aligned} f(z)\in C_p(\gamma,\delta+1,\lambda,\ell) &\Leftrightarrow \frac{zf'(z)}{p}\in S_p^*(\gamma,\delta+1,\lambda,\ell)\\ &\Leftrightarrow L_c\left(\frac{zf'(z)}{p}\right)\in S_p^*(\gamma,\delta+1,\lambda,\ell)\\ &\Leftrightarrow \frac{z}{p}(L_cf(z))'\in S_p^*(\gamma,\delta+1,\lambda,\ell)\\ &\Leftrightarrow L_cf(z)\in C_p(\gamma,\delta+1,\lambda,\ell)\end{aligned} \quad (3.8)$$

Theorem 3.3

Let $c>-\gamma, 0\leq\gamma<p$. If $f(z)\in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$, then

$$L_cf(z)\in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$$

Proof: let $f(z)\in k_p(\beta,\gamma,\delta+1,\ell)$. Then by definition, there exists a function $g(z)\in S_p^*(\gamma,\delta+1,\lambda,\ell)$ such that

$$Re\left\{\frac{z(I_p(\delta,\lambda,\ell)f(z))'}{I_p(\delta,\lambda,\ell)g(z)}\right\}>\beta \qquad (z\in U). \quad (3.9)$$

Then

$$\frac{z\left(I_p(\delta,\lambda,\ell)L_cf(z)\right)'}{I_p(\delta,\lambda,\ell)l_cg(z)}-\beta=(p-\beta)h(z) \quad (3.10)$$

where $h(z)=c_1z+c_1z^2+\cdots$. From (3.3) and (3.10), we have

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)g(z)}=\frac{I_p(\delta+1,\lambda,\ell)(zf'(z))}{I_p(\delta+1,\lambda,\ell)g(z)}$$

$$=\frac{z\left(I_p(\delta,\lambda,\ell)L_c(zf'(z))\right)'+\frac{1}{\lambda}[p(1-\lambda)+\ell]I_p(\delta,\lambda,\ell)L_c(zf'(z))}{z\left(I_p(\delta,\lambda,\ell)L_cg(z)\right)'+\frac{1}{\lambda}[p(1-\lambda)+\ell]I_p(\delta,\lambda,\ell)L_cg(z)}$$

$$=\frac{\frac{z\left(I_p(\delta,\lambda,\ell)L_c(zf'(z))\right)'}{I_p(\delta,\lambda,\ell)L_c(g(z))}+\frac{[p(1-\lambda)+\ell)]I_p(\delta,\lambda,\ell)L_c(zf'(z))}{\lambda I_p(\delta,\lambda,\ell)L_c(g(z))}}{\frac{z\left(I_p(\delta,\lambda,\ell)L_cg(z)\right)'}{I_p(\delta,\lambda,\ell)L_c(g(z))}+\frac{1}{\lambda}[p(1-\lambda)+\ell)]} \quad (3.11)$$

since $g(z)\in S_p^*(\gamma,\delta+1,\lambda,\ell)$, then form Theorem 3.1 we have $L_c(g(z))\in S_p^*(\gamma,\delta+1,\lambda,\ell)$. Let

$$\frac{z\left(I_p(\delta,\lambda,\ell)L_c(g(z))\right)'}{I_p(\delta,\lambda,\ell)l_cg(z)}=(p-\gamma)H(z)+\gamma, \quad (3.12)$$

sjst.scst.edu.ly
Surman Journal for Science and Technology
ISSN: Online (2790-5721) - Print (2790-5713)
مجلة صرمان للعلوم والتقنية
Vol6, No.2, Jun. – Dec. 2024
Pages: 401 ~ 410

Where $Re\ H(z) > 0$. Using (3.11), we have

$$\frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{I_p(\delta+1,\lambda,\ell)g(z)} = \frac{\frac{z(I_p(\delta,\lambda,\ell)L_c(zf'(z)))'}{I_p(\delta,\lambda,\ell)L_c(g(z))}+\frac{1}{\lambda}[p(1-\lambda)+\ell]\big(\beta+(p-\beta)\big)h(z)}{(p-\gamma)H(z)+\gamma++\frac{1}{\lambda}[p(1-\lambda)+\ell]} \quad (3.13)$$

Also, (3.10) can be written as

$$z\big(I_p(\delta,\lambda,\ell)L_cf(z)\big)' = I_p(\delta,\lambda,\ell)L_c\big(g(z)\big)[\beta+(p-\beta)h(z)] \quad (3.14)$$

Differentiating both sides, we have

$$z[z(I_P(\delta,\lambda,\ell)L_Cf(z)']'$$
$$= z\left(I_p(\delta,\lambda,\ell)L_Cg(z)\right)'[\beta+(p-\beta)h(z)]+(p-\beta)zh'(z)I_p(\delta,\lambda,\ell)L_Cg(z) \quad (3.15)$$

or

$$\frac{z\left[z\big(I_p(\delta,\lambda,\ell)L_Cf(z)\big)'\right]'}{I_p(\delta,\lambda,\ell)L_Cg(z)} = \frac{z\Big(z\big(I_p(\delta,\lambda,\ell)L_C(zf'(z))\big)\Big)'}{I_p(\delta,\lambda,\ell)L_Cg(z)} \quad (3.16)$$
$$=(p+\beta)zh'(z)+[\beta+(p-\beta)h(z)][\gamma+(p-\gamma)H(z)].$$

Now, from (3.13) we have

$$\frac{z\big(I_p(\delta+1,\lambda,\ell)f(z)\big)'}{I_p(\delta+1,\lambda,\ell)g(z)}-\beta = (p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\gamma)H(z)+\gamma+\frac{1}{\lambda}[p(1-\lambda)+\ell]} \quad (3.17)$$

Taking $h(z) = u$ and $zh'(z) = v$, we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = (p-\beta)u+\frac{(p-\beta)v}{(p-\gamma)H(z)+\gamma+\frac{1}{\lambda}[p(1-\lambda)+\ell]} \quad (3.18)$$

It is easy to see that the function $\varphi(u,v)$ satisfies the conditions (i) and (ii) of Lemma 2.2 in $D = \mathbb{C}\times\mathbb{C}$. To verify the condition (iii), proceed as follows:

$$Re\ \varphi(iu_2,v_1) = \frac{(p-\beta)v_1\left[\gamma+(p-\gamma)h_1(x,y)+\frac{1}{\lambda}(p(1-\lambda)+\ell)\right]}{\left[(p-\gamma)h_1(x,y)+\gamma+\frac{1}{\lambda}[[p(1-\lambda]+\ell]\right]^2+[(p-\gamma)h_2(x,y)]^2} \quad (3.19)$$

where $H(Z) = h_1(x,y)+ih_2(x,y), h_1(x,y)$ and $h_2(x,y)$ being the functions of x and y and $Re\ H(z) = h_1(x,y) > 0$.

By putting $v_1 \le \frac{-1}{2}(1+u_2^2)$, we obtain

$$Re\ (iu_2,v_1) \le \frac{-(p-\beta)(1+u_2)^2\left[(p-\gamma)h_1(x,y)+\gamma+\frac{1}{\lambda}(p(1-\lambda)+\ell)\right]}{2\left[\left[(p-\gamma)h_1(x,y)+\gamma+\frac{1}{\lambda}(p(1-\lambda)+\ell)\right]^2+[(p-\gamma)h_2(x,y)]^2\right]} \quad (3.20)$$

Hence, $Re\ h(z) > 0, (z \in U)$ and $L_cf(z) \in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$.
Thus, we have $L_Cf(z) \in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$▪

Theorem 3.4 Let $c > -\gamma, 0 \le \gamma < p$. If $f(z) \in k_p^*(\beta,\gamma,\delta+1,\lambda,\ell)$, then

$$L_cf(z) \in k_p^*(\beta,\gamma,\delta+1,\lambda,\ell)$$

Proof:

Consider the following:

$$f(z) \in k_p^*(\beta,\gamma,\delta+1,\lambda,\ell) \Leftrightarrow zf'(z) \in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$$
$$\Rightarrow L_c(zf'(z) \in k_p(\beta,\gamma,\delta+1,\lambda,\ell)$$

$$\Leftrightarrow z(L_c f(z))' \in k_p(\beta, \gamma, \delta + 1, \lambda, \ell)$$
$$\Leftrightarrow L_c f(z) \in k_p^*(\beta, \gamma, \delta + 1, \lambda, \ell) \quad \blacksquare$$

4. Conclusion.

In this paper we investigate and proofed all properties of Catas operator with some classes of analytic functions in U.

Properties of combining Catas operator with integral operator also studied and proofed for these classes.

5.References

[1] A. Catas, On certain classes of p – valent functions defined by multiplier transformations, in: Proceeding of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, August 2007.

[2] E. Deniz, On p- valently Close - to – Convex, Starlike and Convex Functions, Hacettepe Journal of Mathematics and Statistics Volume 41 (5) (2012), 635 – 642.

[3] M. K. Aouf, "On a class of p-valent close-to-convex functions of order β and α," *International Journal of Mathematics and Mathematical Sciences*, vol. 11, pp. 259-266, 1988.

[4] I. S. Jack, "Functions starlike and convex of order α," *Journal of the London Mathematical Society*, vol. 3, pp. 469-474, 1971.

[5] S. S. Miller and P. T. Mocanu, Subordination of differential Superordination, Complex Variables Theory App. 48(2003), no. 10. 815- 826.

[6] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429-446, 1969.

[7] R. J. Libera, "Some class of regular univalent functions," *Proceeding of the American Mathematical Society*, vol. 16, pp. 755-758, 1965.