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Bifurcation and Stability of Reaction Diffusion Equations

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Abstract.

The abstract theory applies to examples of algebraic equations and nonlinear boundary value problems are discussed, including systems of reaction diffusion equations. The link s between bifurcation and stability showing are also investigated, the study discusses how increasing a diffusion coefficient in a reaction-diffusion system can lead to bifurcation and also to a stable solution becoming unstable.

Keywords: Nonlinear Boundary value problems, systems of reaction diffusion equations, Stability, Bifurcation.

1 INTROCUCTION

Bifurcation phenomena arise in many parts of mathematical physics and an understanding of their nature is of practical as well as theoretical importance[1], [2] and [3]. Reaction–diffusion systems are mathematical models which explain how the concentration of one or more substances distributed in space changes under the influence of two processes, local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space [4],[5], [6] and [7]. This paper draws on and derives support from the studies mentioned above and investigates how stability and bifurcation are linked as both depend on the signs of eigenvalues. This study discusses the stability of solutions and reaction diffusion system, and also, it discusses the surprising fact that increasing the amount of diffusion in a system can cause a stable solution to become unstable. Finally, the paper establishes a connection between stability and the direction of bifurcation for semi linear boundary value problems. The researchers consider the stability of solutions of systems of ODE's and diffusion equations and study how the stability is linked to bifurcation

2 REACTION DIFFUSION EQUATION

Let us consider the following equation

$$u_t = Au, \qquad u(0) = u_0 \qquad (1.1)$$

Where $A: X \to Y$ and X and Y are Banach spaces with $X \subseteq Y$. If $X = Y = \mathbb{R}^n$, we have a system of ODE's. If $X = \{u \in C^2([0,1]), u(0) = 0 = u(1)\}$ and $Y = C([0,1])$ we have a reaction diffusion equation. The solution of (1.1) is denoted by $u(t, u_0)$. An equilibrium $\bar{u}$ is stable if all solutions which start near $\bar{u}$ stay nearby.

Moreover, if all nearby solutions converge to $\bar{u}$ as $t \to \infty$,then $\bar{u}$ is asymptotically stable [6], [8] and [9]. Let's give definitions as follow:

An equilibrium solutions $\bar{u}$ is stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every u_0 for which $|u_0 - \bar{u}| < \delta$, the solution $u(t, u_0)$ satisfies $|u(t, u_0) - \bar{u}| < \varepsilon$ for all $t \geq 0$. The equilibrium solution $\bar{u}$ is asymptotically stable if it is stable and there exists $\delta > 0$ such that $|u(t, u_0) - \bar{u}| \to 0$ as $t \to \infty$ for all u_0 for which $|u_0 - \bar{u}| < \delta$ [10].

3 LINEAR AND NONLINEAR SYSTEMS

Let us consider the linear system

$$u_t = Lu \qquad (1.2)$$

Where L is linear with eigenvalues $\lambda_1 < \lambda_2 < \cdots$ and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$ such that $\{\phi_1, \phi_2, \ldots\}$ form a basis for Y. To show that $u = 0$ is stable if all eigenvalues of L has negative real part. Suppose that the general solution of equation (1.2) can be written

$$u(t) = \sum_{n=1}^{\infty} C_n(t)\phi_n \qquad (1.3)$$

It is needed to find $C_n(t)$. By differentiating the equation (1.3), then, we obtain that

$$\left(\sum_{n=1}^{\infty} C_n(t)\phi_n\right)' = \sum_{n=1}^{\infty} C'_n(t)\phi_n = L\left(\sum_{n=1}^{\infty} C_n(t)\phi_n\right) = \sum_{n=1}^{\infty} C_n(t)L\phi_n = \sum_{n=1}^{\infty} \lambda_n C_n(t)\phi_n$$

Hence, it is required

$$C'_n(t) = \lambda_n C_n(t)$$

And so, we get $\lambda_n C_n(t) = Ae^{\lambda_n t}$, where A is a constant. Thus, we obtain the solution

$$u(t) = \sum_{n=1}^{\infty} Ae^{\lambda_n t}\phi_n$$

Zero solution is stable. It can be shown similarly that if L has a positive eigenvalue then $u = 0$ is unstable. Let us consider nonlinear equation $u_t = F(u)$. Since $u \longrightarrow DF(0)u$ is the best linear approximation to $F(u)$ is not surprising that it can be proved that if $u = 0$ is a stable solution of $u_t = DF(u)$, i.e., if all the eigenvalues of $DF(0)u$ have negative real part, then $u = 0$ is a stable solution of $u_t = DF(u)$. On the other hand, if $DF(0)u$ has a positive eigenvalue then $u = 0$ is an unstable solution. **Example 1**

let us consider the nonlinear equation

$$u_t = \lambda u - cu^2 \text{ where } c \in R \text{ and } c \neq 0.$$

Hence,

$$F(u) = \lambda u - cu^2.$$

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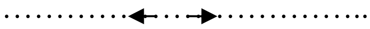
Calculating the Fréchet derivative, we get that

$$F'(u)h = (\lambda - 2cu)h.$$

Hence, $F'(0)h = \lambda h$ and so, if $h \neq 0$, $F'(0)h = \mu h \Leftrightarrow \lambda h = \mu h \Leftrightarrow \lambda = \mu$. Hence $u = 0$ is a stable when $\lambda < 0$ and $u = 0$ is an unstable when $\lambda > 0$. This confirms the conclusion that we reach from the phase planes Figure 1.1

Phase plane when $\lambda < 0$



phase plane when $\lambda > 0$

Figure1.1: phase planes when $u = 0$ is a stable if $\lambda < 0$ and unstable if $\lambda > 0$

Example 2

Let us consider the parabolic equation

$$u_t = u_{xx} + \lambda f(u) \text{ for } 0 < x < 1,\ t > 0; u(0,t) = 0 = u(1,t)$$

The equation above can be written as

$u_t = F(\lambda, u)$ where $F: R \times X \longrightarrow C([0,1])$ with $X = \{u \in C^2([0,1]): u(0) = 0 = u(1)\}$ is defined as

$$F(\lambda, u) = u_{xx} + \lambda f(u).$$

Calculating the Fréchet derivative, we obtain

$$F_u(\lambda, u)h = h_{xx} + \lambda h f'(u)$$

and so when $u = 0$

$$F_u(\lambda, 0)h = h_{xx} + \lambda h f'(0).$$

Thus, the eigenvalues μ of $F_u(\lambda, 0)$ are values of μ such that

$$h_{xx} + \lambda h f'(0) = \mu h \text{ with } h(0) = 0 = h(1) \qquad (1.4)$$

The equation (1.4) has non-zero solutions and it can be written as

$$-h_{xx} = (\lambda f'(0) - \mu)h \text{ with } h(0) = 0 = h(1).$$

Let us assume

$$\gamma = \lambda f'(0) - \mu.$$

Then, we require that

$$-h_{xx} = \gamma h \text{ with } h(0) = 0 = h(1)$$

The eigenvalues $\gamma = \pi^2, 4\pi^2, \ldots$, corresponding to eigenfunctions $\phi_1 = \sin(\pi x)$, $\phi_2 = \sin(2\pi x)$,........, hence, we obtain non zero solution of the equation (1.4) when $f'(0) - \mu = n^2\pi^2$ for some n, for example $\mu = \lambda f'(0) - n^2\pi^2$. Hence $u = 0$ is a stable equilibrium point, if

$$\lambda f'(0) - n^2\pi^2 < 0 \Rightarrow \lambda < \frac{n^2\pi^2}{f'(0)} \text{ for all } n \Rightarrow \lambda < \frac{\pi^2}{f'(0)}.$$

If $\lambda > \frac{\pi^2}{f'(0)}$. Then, the equation (1.4) has a positive eigenvalue and so $u = 0$ is an unstable equilibrium point.

4 STABILITY OF THE REACTION DIFFUSION SYSTEMS.

This section is essentially concerned with stability properties of uniform state solutions to systems of reaction diffusion equations. we consider the system of n reaction diffusion equations.

$$\frac{\partial u}{\partial t} = D\Delta u + F(u) \quad on \ \Omega, \ \frac{\partial u}{dn} = 0 \ on \ \ \partial\Omega \qquad (1.5)$$

Where Ω is a bounded region in R^n, $u \in R^n$, D is an $n \times n$ diagonal matrix and $F: R^n \longrightarrow R^n$. Then, solutions of the corresponding systems of ODE's

$$u_t = F(u) \ u \in R^n \qquad (1.6)$$

Are also solutions of the reaction diffusion system. We would expect the addition of diffusion to make solutions more likely to be stable, we might expect that $u = 0$ is a stable solution of (1.5) whenever is a stable solution of (1.6). We show that this true for the case of single equation.
Let us assume $u = 0$ is a stable of the single equation $u_t = Au$, where A is a negative constant. Let us consider the single reaction diffusion equation $u_t = d\Delta + Au$, this equation more likely to be stable to systems of reaction diffusion equations. The stability of the solution $u = 0$ is determined by the sign of the eigenvalues

$$d\Delta u + Au = \mu u \qquad (1.7)$$

$$-d\Delta u - (A - \mu)u = 0 \qquad (1.8)$$

Hence μ is an eigenvalue of the equation (1.8), if and only if $A - \mu$ is an eigenvalue of $-d\Delta u$ with Neumann boundary condition i.e., $A - \mu$=0, μ_1, μ_2, i.e.,$\mu = A$, $A - \mu_1$. Since $A < 0$ then, all eigenvalues of the equation (1.8) <0 and so, the equilibrium solution is stable. We show that this is not true for the case of a general system as follows. Let us assume $u - 0$ is a stable solution of general linear system $u_t = Au$. Hence all the eigenvalues of A have negative part. We now consider the general reaction diffusion system

$$u_t = D\Delta u + Au$$

The stability the zero solution for this system is determined by of the eigenvalues μ of

$$D\Delta u + Au = \mu u$$

$$-D\Delta u - Au = -\mu u \qquad (1.9)$$

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Non zero solutions of (1.9) occur when $-\mu$ is an eigenvalue of one of the matrices $\mu_i D - A$ i.e μ is an eigenvalue of one of the matrices $-\mu_i D + A$.

Although all the eigenvalues of $-\mu_i D$ and of A have negative real part, it is still possible that $-\mu_i D + A$ has a positive eigenvalue and so (1.9) can have a positive eigenvalue. To illustrate this case, let us give an example. The matrix

$$A = \begin{bmatrix} -10 & 5 \\ -5 & 2 \end{bmatrix}$$

has negative eigenvalues, but by choosing $\mu_1 = 1$ and $D = \begin{bmatrix} 30 & 0 \\ 0 & 1 \end{bmatrix}$, therefore, it is obtained that.

$$A - \mu_1 D = \begin{bmatrix} -10 - 30 & 5 \\ -5 & 2 - 1 \end{bmatrix},$$

This has one positive and one negative eigenvalue. Hence in this case $\mu = 0$ is a stable solution of the system of ODE's $u_t = Au$ but an unstable solution of the reaction system $u_t = D\Delta U + Au$.

5 STABILITY OF THE BIFURCATING SOLUTIONS

By considering the following problem

$$-\mu_{xx} = \lambda f(u), \quad u(0) = 0 = u(1) \qquad (1.10)$$

where $f(0) = 0$ and $f'(0) > 0$. According to the Crandall and Rabinowitz theorem, then there is a nontrivial continuously differentiable curve through $(\lambda_1, 0)$, of the form of $(\lambda(s), u(s))$, where $u(s) = s\phi_1 + s\psi(s)$, and s is small number and ϕ_1 is the first eigenvalue of $-\mu_{xx}$ with zero boundary conditions i.e., $\phi_1(x) = \sin(\pi x)$. In order to investigate the stability of the solution $u(s)$. By linearization of the equation (1.10), then

$$w_{xx} + \lambda f'(u(s))w = \mu w \qquad (1.11)$$

The solution $u(s)$ will be stable (unstable) if the eigenvalue μ corresponding to the positive eigenfunction $w < 0$ (> 0). Substituting the solution $(\lambda(s), u(s))$ in the equation (1.10). Then,

$$-u_{xx}(s) = \lambda(s) f(u(s)) \qquad (*)$$

Differentiating this equation with respect to s yield the new system

$$-u_{xxs}(s) = \lambda'(s) f(u(s)) + \lambda(s) f'(u(s)) u_s(s)$$

Let us consider $\varphi = u_s(s)$. It is clear that by $u(s) = s\phi_1 + s\psi(s)$, and so $u_s(0) = \phi_1 > 0$, and $u_s(s) > 0$ for s close to zero. Then,

$$-\varphi_{xx}(s) = \lambda'(s) f(u(s)) + \lambda(s) f'(u(s))\varphi \qquad (1.12)$$

We multiply (1.11) by φ and (1.12) by w, and by adding new both equations, and integrate to obtain

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$$\int (w_{xx}\varphi - \varphi_{xx}w)dx = \mu \int w\varphi dx + \lambda'(s) \int f(u)\, wdx$$

And also, by using integration parts, the new results can be written as:

$$\int (w_{xx}\varphi - \varphi_{xx}w)dx = 0$$

Hence,

$$\mu = \frac{-\lambda'(s) \int f(u)\, wdx}{\int w\varphi dx} \qquad (1.13)$$

We have that $u(s) > 0$ for small positive s and $u(s) < 0$ for small negative s. Hence $f(u) > 0$ (<0) for s small and positive (negative), and so$\int f(u)\, wdx$ satisfies the same sign property. As $\int w\varphi dx > 0$, it is easily to determine the sign of μ in terms of the signs $\lambda'(s)$. By looking at $\lambda'(0) > 0$. If $s > 0$, $\lambda(s) > \lambda_1$ and is $s < 0$, $\lambda(s) < \lambda_1$, and so the bifurcation diagram is as in the Figure 1.2

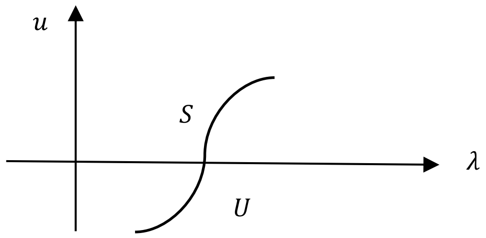

Figure 1.2: Bifurcation diagram for $u(s)$

If $s > 0$, $\int f(u)\, wdx > 0$ and so $\mu < 0$. Hence $u(s)$ is a stable solution. On the other hand, if $s < 0$, $\int f(u)\, wdx < 0$ and so $\mu < 0$. Hence $u(s)$ is an unstable solution. Let us now look at $\lambda'(0) < 0$. Then, if $s > 0$, $\lambda(s) < \lambda_1$ and if $s < 0$, $\lambda(s) > \lambda_1$, To see the bifurcation diagram look at the Figure 1.3 for this case.

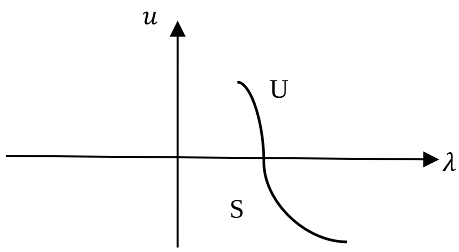

Figure 1.3: Bifurcation diagram when $\mu < 0$

If $s > 0$, $\int f(u)\, wdx > 0$ and so $\mu > 0$. Hence, $u(s)$ is an unstable solution. While if $s < 0$, $\int f(u)\, wdx < 0$ and so $\mu > 0$. Hence, $u(s)$ is a stable solution. Finally, by looking at the case $\lambda'(0) = 0$. Let us first assume $\lambda''(0) > 0$. Hence λ has local minimum at $s = 0$ and so we have bifurcation diagram of the form

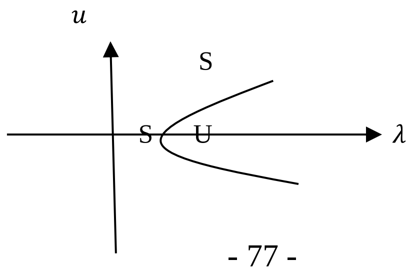

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Figure 1.4: Bifurcation diagram when $\mu > 0$

Let us investigate the stability of $u(s)$ for $s > 0$. Then, $\lambda'(s) > 0$ and $\int f(u)\, wdx > 0$ and so

$$\mu = \frac{-\lambda'(s) \int f(u)\, wdx}{\int w\varphi dx} < 0$$

and so $u(s)$ is a stable solution. On the other hand, for $s < 0$. We have $\lambda'(s) < 0$ and $\int f(u)\, wdx < 0$ and so again

$$\mu = \frac{-\lambda'(s) \int f(u)\, wdx}{\int w\varphi dx} < 0$$

and so $u(s)$ is a stable solution. By considering the case $\lambda''(0) < 0$. Then, λ has local maximum at $s = 0$ and so, the bifurcation diagram is shown in the Figure 1.5:

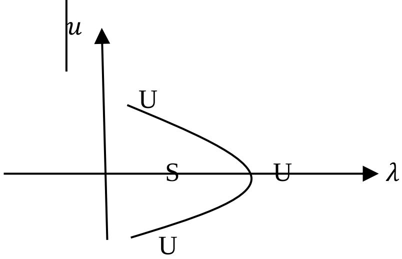

Figure 1.5: Bifurcation diagram of $u(s)$

Let us now investigate the stability of $u(s)$ for $s > 0$. Then $\lambda'(s) < 0$ and so the least eigenvalue of the problem linearized about $u(s)$ is

$$\mu = \frac{-\lambda'(s) \int f(u)\, wdx}{\int w\varphi dx} > 0$$

and so $u(s)$ is an unstable solution. On the other hand if $s < 0$, we have $\lambda'(s) > 0$ and so

$$\mu = \frac{-\lambda'(s) \int f(u)\, wdx}{\int w\varphi dx} > 0$$

and so $u(s)$ is again an unstable solution. It is needed to show how our results apply by giving the following example

$$-u_{xx} = \lambda \sin(u), \quad u(0) = 0 = u(1)$$

It is given $f(u) = \sin(u)$. Then, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$. Thus, it is obtained that $\lambda'(0) = 0$ and $\lambda''(0) > 0$, and so the bifurcation diagram is as in the Figure 1.4

6 CONCLUSIONS

In this paper, we have discussed the stability of the solutions showing how bifurcation and stability are linked, how a uniform solution can become unstable as diffusion (the bifurcation parameter) increases and finally for boundary value problems establish a connection between the stability and direction of bifurcation of bifurcating solutions.

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