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Show some Methods for Computing the Drazin Inverse

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Abstract
The Drazin inverse of a matrix A is a matrix A^Dthat satisfies the condition $A^D A A^D = A^D$, $AA^D = A^D A$, and $A^{k+1} A^D = A^k$. In general, Drazin inverses are exist and unique where A is singular or even rectangular, while if A is non-singular square matrix, then A^D reduces to the usual inverse of A and denoted by A^{-1}.

Keywords: Drazin Inverse, Singular Matrices, Partitioned Matrices

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Introduction

Let A be complex square matrix. The Drazin inverse [1,5] of Ais the unique matrix A^Dsatisfies the following conditions: $A^D A A^D = A^D$, $AA^D = A^D A$, $A^{k+1} A^D = A^k$.

where Ind(A) = k is called the index of A, it is the smallest non-negative such that rank (A^k) =rank (A^{k+1}). We know that A^D always exists and $A^D = A^{-1}$ for Ind(A) =0 Properties the Drazin inverse can be found in [1,5].

In 1979, Campbell and Meyer[5] proposed an open problem to find an explicit representation for the Drazin inverse of partitioned matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$with A_1 and A_4square in terms of $A_1, A_2, A_3$and A_4. But because of the difficulty of this problem, until now it has not been solved yet even for the case $A_4 = 0$. There are many representations for the Drazin inverse of special 2×2 partitioned matrix but under special conditions, some of them can be found in [4]. For example, in 2010, Bu and Zhang [4] gave explicit representation for the Drazin inverse of $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ under the conditions $A_1 A_2 A_3 = 0$ and either $A_4 A_3 = 0$ or $A_2 A_4 = 0$.

In this article, we give the representation of the Drazin inverse for $\begin{bmatrix} A_1 & A_2 \\ O & A_4 \end{bmatrix}$ with A_1 and A_3 square and singular under the conditions that rank(A_1)= rank(A_3)=1, Trac(A_1) $\neq 0$ and Trac(A_3) $\neq 0$, and some corollaries, and then we give the representation of the Drazin inverse for $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 is square and non-singular under the conditions rank(A_1)= rank($\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$) and $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$= $\begin{bmatrix} I \\ P \end{bmatrix} A_1 [I \quad Q]$ where $P = A_3 A_1^{-1}$ and $Q =$

$A_1^{-1}A_2$. Also, we give the the representation of the Drazin inverse for $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 square and singular under the conditions that rank(A_1)= rank($\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$)=1, Trace($\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$) $\neq 0$.

Throughout this paper, $\mathbb{C}^{n\times n}$ is the set of $n \times n$ complex matrices. The identity matrix of $\mathbb{C}^{n\times n}$ is denoted by I or I_n. The trace of a matrix A is denoted by Tr (A). The conjugate transpose of A is denoted by A^*.

Definition: Let A $\in \mathbb{C}^{n\times n}$, with Ind(A) = k, then the Drazin inverse of A is defined to be the unique matrix A^D such that

$A^D A A^D = A^D$

$AA^D = A^D A$, and

$A^{k+1} A^D = A^k$.

Theorem 1.[4] Let A $\in \mathbb{C}^{nxn}$. if $A^D \in \mathbb{C}^{nxn}$ exists, then it is unique.

Proof. Suppose X and y are both Drazin inverse of A. then

$X = XAX = XAXAX = \ldots\ldots = X(AX)^k = A^k X^{k+1}$

$= A^k(YA)X^{k+1} = \ldots\ldots = A^k(YA)^{k+1}X^{k+1}$

$= Y^{k+1}A^{2k+1}X^{k+1}$

$= Y^{k+1}A^k (XA)^{k+1} = \ldots\ldots = Y^{k+1}A^k (XA)$

$= Y^{k+1}A^k = (YA)^k Y = \ldots\ldots = YAYAY = YAY = Y.$

Theorem2. [5] **If** A $\in \mathbb{C}^{n\times n}$, is such that rank(A) = 1, then $A^D = \frac{1}{[Tr(A)]^2}$ A **when** *Tr(A)* $\neq 0$ ***and*** $A^D = 0$ when *Tr(A) = 0.*

Example: If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \mathbb{C}^{2\times 2}$, rank(A) = 1, and *Tr*(A) = 1+1=2 $\neq 0$, then $A^D = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Lemma. If A $\in \mathbb{C}^{n\times n}$, is such that rank(A) = 1 and $A^D = \frac{1}{[Tr(A)]^2}$ A *where Tr(A)* $\neq 0$, then $AA^D - K = I$, where

$$K = \begin{bmatrix} \frac{-(Tr(A)-a_{11})}{Tr(A)} & \frac{a_{12}}{Tr(A)} & \cdots & \frac{a_{1n}}{Tr(A)} \\ \frac{a_{21}}{Tr(A)} & \frac{-(Tr(A)-a_{22})}{Tr(A)} & \cdots & \frac{a_{2n}}{Tr(A)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A)} & \frac{a_{n2}}{Tr(A)} & \cdots & \frac{-(Tr(A)-a_{nn})}{Tr(A)} \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Proof. Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix},$$

Such that rank(A) = 1 and Tr(A)$\neq$ 0, where Tr(A)=$a_{11}+a_{22}+\cdots+a_{nn}$.

$$A^D=\frac{1}{(Tr(A))^2}A.$$

$$AA^D=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix} \frac{-(Tr(A)-a_{11})}{[Tr(A)]^2} & \frac{a_{12}}{[Tr(A)]^2} & \cdots & \frac{a_{1n}}{[Tr(A)]^2}\\ \frac{a_{21}}{[Tr(A)]^2} & \frac{-(Tr(A)-a_{22})}{[Tr(A)]^2} & \cdots & \frac{a_{2n}}{[Tr(A)]^2}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{a_{n1}}{[Tr(A)]^2} & \frac{a_{n2}}{[Tr(A)]^2} & \cdots & \frac{-(Tr(A)-a_{nn})}{[Tr(A)]^2}\end{bmatrix}$$

$$=\begin{bmatrix} \frac{\sum_{j=1}^n a_{1j}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{1j}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{1j}a_{jn}}{[Tr(A)]^2}\\ \frac{\sum_{j=1}^n a_{2j}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{2j}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{2j}a_{jn}}{[Tr(A)]^2}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\sum_{j=1}^n a_{nj}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{nj}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{nj}a_{jn}}{[Tr(A)]^2}\end{bmatrix}.$$

$AA^D-K=$

$$\begin{bmatrix} \frac{\sum_{j=1}^n a_{1j}a_{j1}+(Tr(A))^2-\sum_{j=1}^n a_{11}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{1j}a_{j2}-\sum_{j=1}^n a_{12}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{1j}a_{jn}-\sum_{j=1}^n a_{1n}a_{jj}}{[Tr(A)]^2}\\ \frac{\sum_{j=1}^n a_{2j}a_{j1}-\sum_{j=1}^n a_{21}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{2j}a_{j2}+(Tr(A))^2-\sum_{j=1}^n a_{11}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{2j}a_{jn}-\sum_{j=1}^n a_{2n}a_{jj}}{[Tr(A)]^2}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\sum_{j=1}^n a_{nj}a_{j1}-\sum_{j=1}^n a_{nj}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^n a_{nj}a_{j2}-\sum_{j=1}^n a_{n2}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^n a_{nj}a_{jn}+(Tr(A))^2-\sum_{j=1}^n a_{nn}a_{jj}}{[Tr(A)]^2}\end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}=I.$$

Example: Let $A=\begin{bmatrix} 1 & 2 & 3\\ 2 & 4 & 6\\ 1 & 2 & 3\end{bmatrix}$.

Tr(A) = 1+4+3 = 8, $A^D=\frac{1}{8^2}\begin{bmatrix} 1 & 2 & 3\\ 2 & 4 & 6\\ 1 & 2 & 3\end{bmatrix}$, $A\,A^D=\begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{3}{8}\\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4}\\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8}\end{bmatrix}$,

$$A\,A^D - K = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{bmatrix} - \begin{bmatrix} \frac{-7}{8} & \frac{2}{8} & \frac{3}{8} \\ \frac{2}{8} & \frac{-4}{8} & \frac{6}{8} \\ \frac{1}{8} & \frac{2}{8} & \frac{-5}{8} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem3. [5] If $A \in \mathbb{C}^{n\times n}$, is such that $\mathrm{Ind}(A) = k > 0$, then there exists a non-singular matrix p such that

$$A = p \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} p^{-1}$$

Where C is non-singular, and N is nilpotent of index k.

Furthermore, if P, C *and* N are any matrices satisfying the above conditions, then

$$A^D = p \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} p^{-1}.$$

Example: Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$, $Ind(A) = 1$

$$p = \begin{bmatrix} 9 & 1 \\ 0 & 1 \end{bmatrix}, \quad p^{-1} = \begin{bmatrix} \frac{1}{9} & -\frac{1}{9} \\ 0 & 1 \end{bmatrix}, \quad p^{-1}pp^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$$

$$A^D = \begin{bmatrix} 9 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{9} & -\frac{1}{9} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

Theorem4. [2] If $A \in \mathbb{C}^{m\times n}$, then there exists $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$ such that $A = BC$ and $r = \mathrm{rank}(A) = \mathrm{rank}(B) = \mathrm{rank}(C)$, then

$A^D = (BC)^D = C^*(CC^*)^{-1}(B^*B)^{-1}B^*$.

Lemma if $A_1, A_2, A_3,$ *and* A_4 are matrices such that A_1 is square and non-singular and rank $(A_1) = \mathrm{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right)$, then

$$A_4 = A_3 {A_1}^{-1} A_2.$$

Furthermore, if $P = A_3 {A_1}^{-1}$ and $Q = {A_1}^{-1} A_2$ then

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 \begin{bmatrix} I & Q \end{bmatrix}$$

Theorem5. [3] Let $H = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathbb{C}^{n\times n}$,

Be an EP matrix where A_1, A_2, A_3 and A_4 are matrices such that A_1 is square and non-singular and $\mathrm{rank}(A_1) = \mathrm{rank}(H)$, $A_4 = A_3 A_1^{-1} A_2$. If P and Q are any matrices such that.

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 \begin{bmatrix} I & Q \end{bmatrix},$$

Then

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^D = \begin{bmatrix} I \\ Q^* \end{bmatrix} ([\mathrm{I}+P^*P]A_1[\mathrm{I}+\mathrm{Q}\,Q^*])^{-1}[I \quad P^*]$$

Proof: Let $A_1 \in \mathbb{C}^{n\times n}$, $\mathrm{B} = \begin{bmatrix} I_n \\ P \end{bmatrix} A_1$, $\mathrm{C} = [I_n \quad Q]$.

Note that rank(B) = rank(C) = rank(A_1) = rank(H) = n = number of columns of B = number of rows of C. Thus we can apply Theorem4 to get

$$\mathrm{H}^D = (BC)^D = \mathrm{C}^*(\mathrm{CC}^*)^{-1}(\mathrm{B}^*\mathrm{B})^{-1}\mathrm{B}^*.$$

Since

$$(\mathrm{B}^*\mathrm{B})^{-1}\mathrm{B}^* = [A_1^*(I+P^*P)A_1]^{-1}A_1^*[I \quad P^*]$$
$$= A_1^{-1}(I+P^*P)^{-1}\,[I \quad P^*],$$

And

$$\mathrm{C}^*(\mathrm{CC}^*)^{-1} = \begin{bmatrix} I \\ Q^* \end{bmatrix}(I+QQ^*)^{-1},$$

Then H^D is obtained.

Example: Let $\mathrm{H} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathbb{C}^{4\times 4}$ be an EP matrix, where

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that A_1 is square and non-singular, and rank (A_1) = rank(H) = 2. we see that $\mathrm{A}_4 = \mathrm{A}_3\,\mathrm{A}_1^{-1}\mathrm{A}_2$, where $\mathrm{A}_1^{-1} = \mathrm{A}_1$,

$$\mathrm{P} = A_3\,A_1^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \mathrm{Q} = A_1^{-1}A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\mathrm{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

$$\mathrm{H}^\mathrm{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}.$$

Theorem6. [2] **Let** $H=\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}\in\mathbb{C}^{n\times n}$

Where A_1 is square and singular whit rank (A_1) =rank(*H*) =*1*, *Tr(H)* $\neq$ 0, then

$$H^D=\frac{1}{[Tr(A_1)+Tr(A_4)]^2}H.$$

Example. Let

$$H=\begin{bmatrix}2 & -2 & 0 & 2\\-2 & 2 & 0 & 2\\1 & -1 & 0 & -1\\-4 & 4 & 0 & 4\end{bmatrix}=\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}\in\mathbb{C}^{4\times 4}$$

were

$$A_1=\begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix},\quad A_2=\begin{bmatrix}0 & 2\\0 & 2\end{bmatrix}, A_3=\begin{bmatrix}1 & -1\\-4 & 4\end{bmatrix},\quad A_4=\begin{bmatrix}0 & -1\\0 & 4\end{bmatrix}.$$

Note that A_1 is square and singular, and rank (A_1) = rank(H) =1, *Tr(H)*=2+2+4 = 8$\neq$ 0, then

$$H^D=\frac{1}{[Tr(A_1)+Tr(A_4)]^2}H$$

$$=\frac{1}{[4+4]^2}\begin{bmatrix}2 & -2 & 0 & 2\\-2 & 2 & 0 & 2\\1 & -1 & 0 & -1\\-4 & 4 & 0 & 4\end{bmatrix}$$

$$=\frac{1}{64}\begin{bmatrix}2 & -2 & 0 & 2\\-2 & 2 & 0 & 2\\1 & -1 & 0 & -1\\-4 & 4 & 0 & 4\end{bmatrix}.$$

Theorem7. [2] **Let** $B\in\mathbb{C}^{n\times n}$ is B = $\begin{bmatrix}A_1 & A_2\\0 & A_3\end{bmatrix}$, Where

$$A_1=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\a_{21} & a_{22} & \cdots & a_{2n}\\\cdots & \cdots & \cdots & \cdots\\a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix},\quad A_3=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\b_{21} & b_{22} & \cdots & b_{2n}\\\cdots & \cdots & \cdots & \cdots\\b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix},$$

are square and singular with rank (A_1) = rank (A_3) = 1, Tr $(A_1)\neq 0$ and *Tr* $(A_3)\neq 0$, *Ind* (A_1) $= l_1$ and *Ind* $(A_3) = l_2$, then

$$B^D=\begin{bmatrix} A_1^D & Y \\ 0 & A_3^D \end{bmatrix},$$

were

$$A_1^D=\frac{1}{(Tr(A_1))^2}A_1, \quad A_3^D=\frac{1}{(Tr(A_3))^2}A_3,$$

$$Y=(A_1^D)^2\left[\sum_{i=1}^{n}\left[(A_1^D)^iA_2A_3^i\right]\right](-K_2)+(-K_1)\left[\sum_{i=0}^{n}\left[A_1^iA_2\left(A_3^D\right)^i\right]\right](A_3^D)^2-A_1^DA_2A_3^D = (A_1^D)^2\left[\sum_{i=1}^{l_2-1}\left[(A_1^D)^iA_2A_3^i\right]\right](-K_2)+(-K_1)\left[\sum_{i=0}^{l_1-1}\left[A_1^iA_2\left(A_3^D\right)^i\right]\right](A_3^D)^2-A_1^DA_2A_3^D$$

Were

$$K_1=\begin{bmatrix} \frac{-(Tr(A_1)-a_{11})}{Tr(A_1)} & \frac{a_{12}}{Tr(A_1)} & \cdots & \frac{a_{1n}}{Tr(A_1)} \\ \frac{a_{21}}{Tr(A_1)} & \frac{-(Tr(A_1)-a_{22})}{Tr(A_1)} & \cdots & \frac{a_{2n}}{Tr(A_1)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A_1)} & \frac{a_{n2}}{Tr(A_1)} & \cdots & \frac{-(Tr(A_1)-a_{nn})}{Tr(A_1)} \end{bmatrix}, \text{ and}$$

$$K_2=\begin{bmatrix} \frac{-(Tr(A_3)-a_{11})}{Tr(A_3)} & \frac{a_{12}}{Tr(A_3)} & \cdots & \frac{a_{1n}}{Tr(A_3)} \\ \frac{a_{21}}{Tr(A_3)} & \frac{-(Tr(A_3)-a_{22})}{Tr(A_3)} & \cdots & \frac{a_{2n}}{Tr(A_3)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A_3)} & \frac{a_{n2}}{Tr(A_3)} & \cdots & \frac{-(Tr(A_3)-a_{nn})}{Tr(A_3)} \end{bmatrix}$$

Corollary1. Let $B \in \mathbb{C}^{n\times n}$ is $B=\begin{bmatrix} A_3 & 0 \\ A_2 & A_1 \end{bmatrix}$, Where A_1, A_3 are singular square with $rank(A_1)=rank(A_3)=1$, $Tr(A_1)\neq 0$ and $Tr(A_3)\neq 0$, $Ind(A_1)=l_1$ and $Ind(A_3)=l_2$, then

$$B^D=\begin{bmatrix} A_3^D & 0 \\ Y & A_1^D \end{bmatrix}.$$

Corollary2. Let $B_i \in \mathbb{C}^{n\times n}$ Where A_1 is singular square

$$B_1=\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, B_2=\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}, B_3=\begin{bmatrix} 0 & 0 \\ A_2 & A_1 \end{bmatrix}, B_4=\begin{bmatrix} 0 & A_2 \\ 0 & A_1 \end{bmatrix}$$

Then

$$B_1^D=\begin{bmatrix} A_1^D & Y_1 \\ 0 & 0 \end{bmatrix}, B=\begin{bmatrix} A_1^D & 0 \\ Y_2 & 0 \end{bmatrix}, B_3^D=\begin{bmatrix} 0 & 0 \\ Y_3 & A_1^D \end{bmatrix}, B_4^D=\begin{bmatrix} 0 & Y_4 \\ 0 & A_1^D \end{bmatrix}$$

Were

$$A_1^D=\frac{1}{[Tr(A_1)]^2}A_1, \quad Y_1=\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right)A_2, \quad Y_2=A_2\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right),$$

$$Y_3=\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right)A_2, \quad Y_4=A_2\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right).$$

Corollary3. Let $B \in \mathbb{C}^{n\times n}$, is $B = \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix}$, Where A_1 is singular and square with rank (A_1) = rank (B) = 1, Tr $(A_1) \neq 0$ then

$$B^D = \frac{1}{[Tr(A_1)]^2} B.$$

CONCLUDING REMARKES

It is our hope that this paper will be useful for further study of the Drazin inverse and its applications. For example. will be useful for further study of solving a class of second-order singular differential equations.

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