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Techniques Study for Apply Control on Nonlinear Systems and the Concept of the Sliding Mode Control (SMC)

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الملخص:

المبادئ الأساسية لهذه الورقة هو دراسة بعض التقنيات المستخدمة لتطبيق التحكم على الأنظمة غير الخطية .و يتم تطبيق هذه الطرق على ثلاثة أمثلة غير خطية مختلفة من أجل التحقق من أداء هذه التقنيات. وكذلك تهدف هذه الورقة إلى فهم مفهوم التحكم في وضع الانزلاق وهناك ثلاث مشاكل يجب حلها لهذه الورقة وهي أولاً التحكم في وضع الانزلاق ، الإحماء و الثاني في وضع الانزلاق لمعادلة فان دير بول (VDP) ، والثالث تحكم (MIMO) للذراع المستوي ثنائي الوصلة.

Abstract:

The basic principles of this paper are some techniques that used to apply control on nonlinear systems. These methods were applied to three different nonlinear examples in order to check the performance of these techniques. Moreover, this paper is to understand the concept of the sliding mode control and there are three problems that we should solve for this paper first Sliding Mode Control Warm-Up, second Sliding Mode control for the van der pol Equation (VDP), and third MIMO control of two-link Planar Arm.

Keywords: Sliding mode control (SMC) . Van der pol Equation (VDP) . Multiple input, multiple output (MIMO)

1. Introduction:

The Nonlinear control is the part of control theory which deals with systems that are nonlinear, time-variant, or both. Moreover, the control theory is an interdisciplinary branch of engineering and mathematics that is concerned with the behavior of dynamical systems with inputs, and how to modify the output by changes in the input using feedback, feed forward, or signal filtering. Also, the system to be controlled is called the "plant" , and one way to make the output of a system follow a desired reference signal is to compare the output of the plant to the desired output, and provide feedback to the plant to modify the output to bring it closer to the desired output. Therefore, in this paper to clarification the concept of the sliding mode control for the Sliding Mode control Warm-Up, the Sliding Mode control for the van der pol Equation (VDP), and the MIMO control of a Two-link Planar Arm

2. Sliding Mode Control Warm-Up

In control systems the sliding mode control Warm Up (SMC) is a nonlinear control method that alters the dynamics of a nonlinear system by applying a discontinuous control signal (or more rigorously, a set-valued control signal) that forces the system to "slide" along a cross-section of the system's normal behavior. And, the system differential equation and controller design and the system differential equation as follows:-

$$\dot{x}_1 = x_2 + a x_1 sin x_1 \qquad \& \qquad \dot{x}_2 = b x_1 x_2 + u$$

Where x_1 is the first variable and x_2 is the second variable and u is input, and $sinx_1$ is slip angle for first variable. Also, a and b are unknown constants but we know that $0 \le |a| \le 2$ *and* $1 \le |b| \le 3$ .In this experiment we will apply feedback linearization approach to check the system convergence to zero; and the second approach sliding mode control will be applied as well.

$$\bar{f}(x) = \begin{bmatrix} x_2 + a x_1 sin x_1 \\ b x_1 x_2 \end{bmatrix} \quad ; \quad \bar{g}(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this part, we will deal with the system's parameters a, b as unknown parameters $\hat{a}$ and $\hat{b}$. Also, nominal values will be plugged in for them to come up with a diffeomorphism and a controller such that the system converges to zero. For the last two, the system parameters will be changed to another nominal values to check if the controller still effective with system of not. Now to apply feedback linearization, we need to find a diffeomorphism such that the system will be on chain of integral form.

We need to find $\mathbf{z} = \mathbf{T}(\mathbf{x})$ **statisfying**:

$$\frac{\partial T_1}{\partial x}\overline{g} = 0 \quad , \quad \frac{\partial T_2}{\partial x}\overline{g} \neq 0 \quad and \quad T_2 = \frac{\partial T_1}{\partial x}\overline{f}$$

We can write all conditions:

$$(\mathbf{1}) \quad \frac{\partial T_1}{\partial x}\overline{g} = 0 \Longrightarrow \begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0 \quad \Longrightarrow \quad \frac{\partial T_1}{\partial x_2} = 0 \quad \Longrightarrow \; T_1 = T_1(x_1)$$

$$(\mathbf{2}) \quad \frac{\partial T_2}{\partial x}\overline{g} \neq 0 \Longrightarrow \begin{bmatrix}\frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \neq 0 \; \Longrightarrow \quad \frac{\partial T_2}{\partial x_2} \neq 0 \; \Longrightarrow \; T_2 = T_2(x_1, x_2)$$

$$(\mathbf{3}) \quad T_2 = \frac{\partial T_1}{\partial x}\overline{f} = \begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2}\end{bmatrix}\begin{bmatrix}x_2 + ax_1sinx_1\\ bx_1x_2\end{bmatrix} = \frac{\partial T_1}{\partial x_1}\,(x_2 + ax_1sinx_1)$$

Let's choose that $\quad T_1 = x_1 \Longrightarrow \quad \frac{\partial T_1}{\partial x_2}\overline{g} = 0 \quad \Longrightarrow \quad Satisfies\ condition \qquad [1]$

$$T_2 = \frac{\partial T_1}{\partial x}\overline{f} = \begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2}\end{bmatrix}\begin{bmatrix}x_2 + ax_1sinx_1\\ bx_1x_2\end{bmatrix} = \frac{\partial T_1}{\partial x_1}\,(x_2 + ax_1sinx_1)$$

$$\frac{\partial T_1}{\partial x_1} = 1 \;\Longrightarrow \quad T_2 = (x_2 + ax_1sinx_1) \Longrightarrow Satisfies\ condition \qquad [2]\ and\ [3]$$

So, now we need to satisfy the second condition.

$$\frac{\partial T_2}{\partial x}\bar{g} = \begin{bmatrix}\frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = [ax_1\cos(x_1) + a\sin(x_1) \quad 1]\begin{bmatrix}0\\1\end{bmatrix} = 1 \neq 0$$

Then, we conclude that the system is state feedback linearizable.

To get the controller of the system, first, we have to get the diffeomorphism $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x})$ as follows:

$$z = T(x) = \begin{bmatrix}x_1\\ x_2 + ax_1sinx_1\end{bmatrix} \quad \Longrightarrow \quad \begin{matrix}z_1 = x_1\\ z_2 = x_2 + ax_1sinx_1\end{matrix}$$

Here, we need to find $\mathbf{x} = \mathbf{T}^{-1}(\mathbf{z})$

$$x_1 = z_1$$

$$z_2 = x_2 + ax_1sinx_1 \quad \Longrightarrow \quad z_2 = x_2 + az_1sinz_1 \quad \Longrightarrow \quad x_2 = z_2 - az_1sinz_1$$

Then $x = T^{-1}(z)$ will be $\quad \Longrightarrow \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = T^{-1}(z) = \begin{bmatrix}z_1\\ z_2 - az_1sin\,z_1\end{bmatrix}$

To design SFL controller that regulates z $\Longrightarrow 0$

$\dot{z}_1 = \frac{\partial z_1}{\partial x_1} * \dot{x}_1 + \frac{\partial z_1}{\partial x_2} * \dot{x}_2$

$\dot{z}_1 = 1 * \dot{x}_1 + 0 * \dot{x}_2$

$\dot{z}_1 = x_2 + a x_1 \sin(x_1)$

$\dot{z}_1 = z_2 - a z_1 \sin(z_1) + a z_1 \sin(z_1)$

$\dot{z}_1 = z_2$

$\dot{z}_2 = \frac{\partial z_2}{\partial x_1} * \dot{x}_1 + \frac{\partial z_2}{\partial x_2} * \dot{x}_2$

$\dot{z}_2 = [a x_1 \cos(x_1) + a \sin(x_1)] * [x_2 + a x_1 \sin(x_1)] + 1 * (b x_1 x_2 + u)$

$\dot{z}_2 = [a x_1 x_2 \cos(x_1) + a^2 x_1^2 \cos(x_1) * \sin(x_1) + a x_2 \sin(x_1) + a^2 x_1 \sin(x_1) * \sin(x_1)] +$
$b x_1 x_2 + u$

$\dot{z}_2 = a z_1 [z_2 - a z_1 \sin(z_1)] \cos(z_1) + a^2 z_1^2 \cos(z_1) * \sin(z_1) +$

$a\,[z_2 - z_1 \sin(z_1)] \sin(z_1) + a^2\, z_1 \sin^2(z_1) + b z_1 [z_2 - a z_1 \sin(z_1)] + u$

$\dot{z}_2 = a z_1 z_2 \cos(z_1) - a^2 z_1^2 \sin(z_1) * \cos(z_1) + a^2 z_1^2 \cos(z_1) * \sin(z_1) + a z_2 \sin(z_1) -$
$a^2 z_1 \sin^2(z_1) + a^2 z_1 \sin^2(z_1) + b z_1 z_2 - ab z_1^2 \sin(z_1) + u$

$\dot{z}_2 = a z_1 z_2 \cos(z_1) + a z_2 \sin(z_1) + b z_1 z_2 - ab z_1^2 \sin(z_1) + u$

Then the system will be:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(x) + g(x) u \end{aligned}$$

From $\dot{z}_2$, we can get $f(x)$, $g(x)$, $f(z)$, and $g(z)$.

$f(x) = a x_1 x_2 \cos(x_1) + a^2 x_1^2 \cos(x_1) * \sin(x_1) + a x_2 \sin(x_1) + a^2 x_1 \sin(x_1) * \sin(x_1) +$
$b x_1 x_2$

$g(x) = 1$

Now, in *x*- coordinates, we let the controller be

$u = \frac{1}{g(x)}[-f(x) + v]$

$$u = -[ax_1x_2\cos(x_1) + a^2x_1^2\cos(x_1) * \sin(x_1) + ax_2\sin(x_1) + a^2x_1\sin(x_1) * \sin(x_1) + bx_1x_2] + v$$

$u = \frac{1}{g(z)}[-f(z) + v]$

$f(z) = az_1z_2\cos(z_1) + az_2\sin(z_1) + bz_1z_2 - abz_1^2\sin(z_1)$

$g(z) = 1$

$u = -[az_1z_2\cos(z_1) + az_2\sin(z_1) + bz_1z_2 - abz_1^2\sin(z_1)] + v$

For Simulation and results:

$u = \frac{1}{g(z)}[-f(z) + v] \quad \Longrightarrow \quad v = -(k_1z_1 + k_2z_2)$

$\mathbf{k_1}$ & $\mathbf{k_2}$ have to selected to make the location for system poles in the left hand side

thus the system is state feed back linearizable, where $a = 1$, $b = 1.5$, and $v = -k_1z_1 - k_2z_2$ then the control law u that state-feedback linearizes the system will be:

$u = [-z_1z_2\cos(z_1) - z_2\sin(z_1) - 1.5\, z_1z_2 + 1.5\, z_1^2\sin(z_1) - k_1z_1 - k_2z_2]$

Substituting u in $\dot{z}_2$, we get; $\dot{z}_2 = -k_1\, z_1 - k_2\, z_2$

Where $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are positive and should be chosen to place the system poles the LHP.

Then the state–space for the system will be:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k_1 & -k_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$

By simulating the above controller with MATLAB at $a = \hat{a} = 1$ *and* $b = \hat{b} = 1.5$. when we simulate the controller by $\mathrm{a} = \hat{\mathrm{a}} = 1$ and $\mathrm{b} = \hat{\mathrm{b}} = 1.5$ or by nominal values of *a* and *b* that we used for feedback linearizing controller we

will get the response shown in Figure (1). By using the MATLAB statement [k = place (A, B, P)] to make the system stable, we can find the values of $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$**.**

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $P = [-1 \quad -2]$

So, P is the desired eigenvalues. Then, the values will be; $\boldsymbol{k_1 = 2}$ and $\boldsymbol{k_2 = 3}$. So, the obtained state–space will be:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Here, from the above system we can make that the system becomes linear.

In the simulation part, we simulate the controller u by regulating the states to zero.

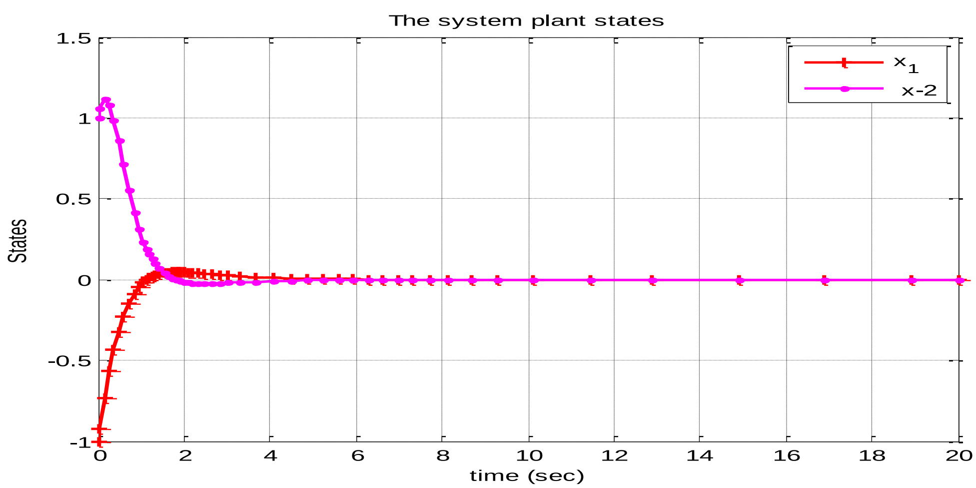

Figure (1) The feedback linearization controller regulates the system states to zero

Here, we can to see clear that both states x_1 and x_2 go to zero and suppose that $\hat{a} \neq a$ and $\hat{b} \neq b$, and that the values of a and b are $a = -1$ and $b = 2$. Using Matlab, plot the phase portrait of the open-loop system (1), and identify the regions where the system exhibits different behaviors. In this part, it was supposed that $a \neq \hat{a}$ and $b \neq \hat{b}$ and by setting a=-1 and b=2 in using MATLAB we will plot the phase portrait of the open-loop system (1), and the regions where the system exhibits different behaviors, also we let $u = 0$ of the open-loop system. By simulating the system, we get the phase portrait of the open-loop system using different initial conditions. The Figure (2) and (3) show the results of the simulation for the states of the plant and the phase portrait plot.

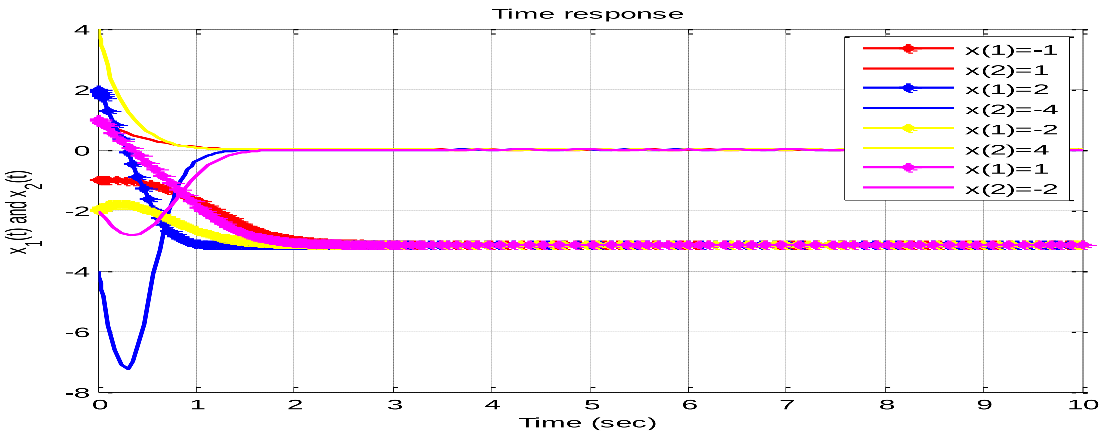

Figure (2): The states vector x of the plant vs time.

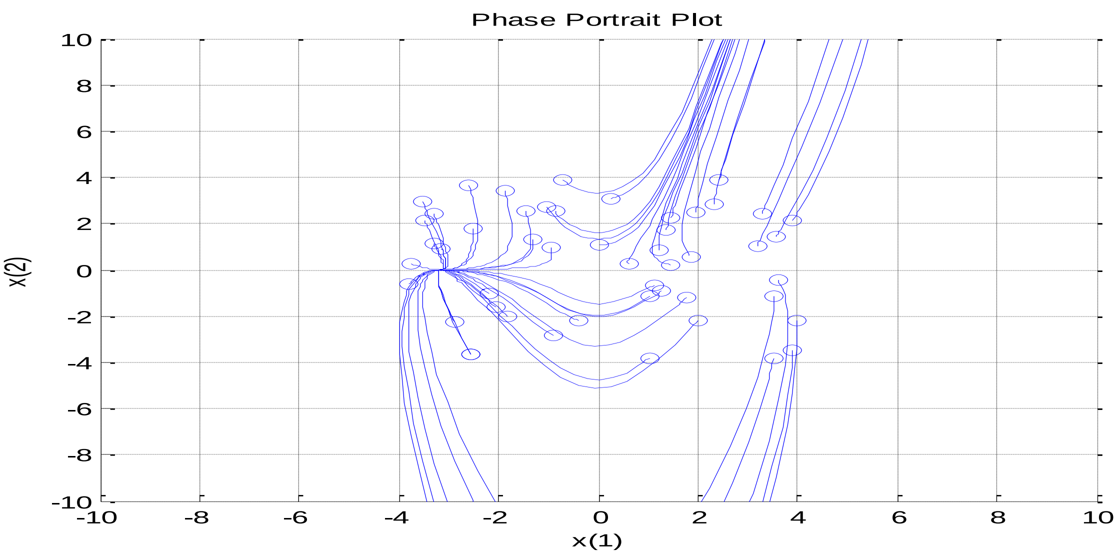

Figure (3): The phase portrait of the system.

For Simulation and results:

The states vector x of the plant vs time sis shown in Figure (2), and the phase portrait of the system is shown in Figure (3).

From Figure (2), we can see that the state x_1 goes to almost -3, and the state x_2 goes to zero. Also, all trajectories in Figure (3) go to almost the points (-3.5, 0) and infinite.

Moreover, we will Explain why it is not possible anymore to use your controller from (a) when $\hat{a} \neq a$ and $\hat{b} \neq b$ and guarantee convergence to zero. In particular pay attention to the diffeomorphism and the controller itself. Verify by setting $a = -1$ and $b = 2$ in your simulation with the same u and diffeomorphism you got in (a). Hint: be sure to use $\hat{a} = 1$ and $\hat{b} = 1.5$ to do your coordinate transformation and to compute your controller, since you assume not to know the real values for control design purposes! At the same time, use the real a =-1 and b = 2 values in your simulation. From part (a), the controller in z-coordinates is:

$$u = \frac{1}{g(z)}[-f(z) + v]$$

$$f(z) = az_1z_2\cos(z_1) + az_2\sin(z_1) + bz_1z_2 - abz_1^2\sin(z_1)$$

$$g(z) = 1$$

$$u = -[az_1z_2\cos(z_1) + az_2\sin(z_1) + bz_1z_2 - abz_1^2\sin(z_1)] + v$$

In this section, we let $a,$ b and v are the same values that we have found in Part (a). Therefore, we put $a = -1$ and $b = 2$ in the simulation part.

$$u = [z_1z_2\cos(z_1) + z_2\sin(z_1) - 2z_1z_2 - 2\,z_1^2\sin(z_1) - 2z_1 - 3z_2]$$

For Simulation and results:

Figure (4) shows the states vector $[x_1 \;\; x_2\,]$ with respect of time. Also, From Figure (4) we can verify that t it is not possible any more to use the controller from (a) to get the states converged to zero. By choosing $a = -1$ and $b = 2$ which the value of a is out of controller limit, the system becomes impossible to use and make the states regulate to zero.

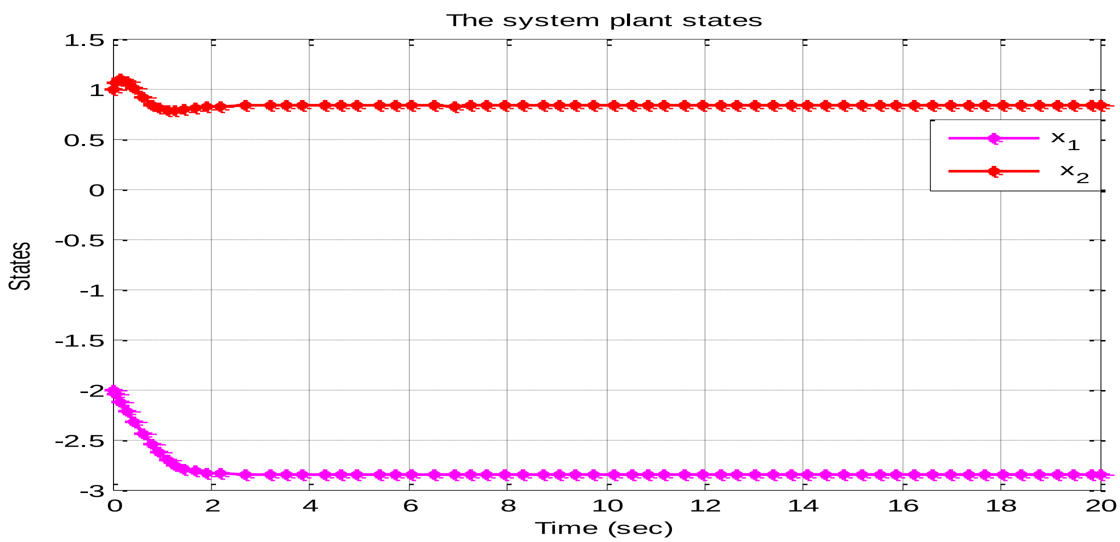

Figure (4): The states vector x of the system

Instead of feedback linearization and we will now implement a sliding mode controller. The approach is to figure out what x_2 should be if we considered it as an input to the $\dot{x}_1$ equation so that x_1 is driven to zero. Design a linear manifold of the form $\boldsymbol{s = cx_1 + x_2}$ that does the job by considering what happens when $s = 0$. Next, analyze the dynamics of s and determine a sliding mode controller that drives s to zero (and consequently to zero).

$$\dot{x}_1 = x_2 + ax_1 sinx_1$$

$$\dot{x}_2 = bx_1x_2 + u$$

The system is considered as:

$$\dot{x} = \bar{f}(x) + \bar{g}(x)u$$

Where $\bar{f}(x) = \begin{bmatrix} x_2 + ax_1 \sin(x_1) \\ bx_1x_2 \end{bmatrix}$ and $\bar{g}(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The sliding mode controller which is $\boldsymbol{s = cx_1 + x_2}$, where $\boldsymbol{c}$ is a positive number.

Let's check the sliding mode by forcing the following:

$$s = cx_1 + x_2 = 0$$

$$x_2 = -cx_1$$

$$v(x) = \frac{1}{2}x_1^2 \quad p.d \quad \Longrightarrow \quad \dot{x}_1 = x_2 + ax_1 sinx_1$$

The derivative of it will be:

$$\dot{v}(x) = x_1 * \dot{x}_1$$

$$\dot{v}(x) = x_1 * [\, x_2 + ax_1 \sin(x_1)]$$

$$\dot{v}(x) = x_1\, x_2 + ax_1^2 \sin(x_1)$$

$$\dot{v}(x) = x_1\, (-cx_1) + ax_1^2 \sin(x_1)$$

$$\dot{v}(x) = -cx_1^2 + ax_1^2 \sin(x_1)$$

As we know that where $0 \leq a \leq 2$, and $\sin(x_1)$ is between the values $+1$ *and* -1. Then,

$$\dot{v}(x) \leq -cx_1^2 + |a|x_1^2$$

where $c > 2$, so we let $= 4$, and we will get:

$\dot{v}(x) \leq -4x_1^2 + |a|x_1^2$

Here, we will use Lyapunov's first method by assuming the following:

$v(s) = \frac{1}{2}s^2$

Also, the derivative of it will be:

$\dot{v}(s) = s * \dot{s}$

$\dot{v}(s) = s * (c[\, x_2 + ax_1 \sin(x_1)] + bx_1x_2 + u)$

$\dot{v}(s) = s\, c[\, x_2 + ax_1 \sin(x_1)] + s\,(bx_1x_2) + s\,u$

$\dot{v}(s) \leq s\, c[\, x_2 + ax_1 \sin(x_1)] + s\,(bx_1x_2) + s(1)u$

For $\dot{s}$, we cancel the known term on the right-hand side, we can do

$u = -4x_2 + v$

Then, dynamics $\dot{s}$ will be:

$\dot{s} = 4x_2 + 4ax_1 \sin(x_1) + bx_1x_2 - 4x_2 + v$

$\dot{s} = 4ax_1 \sin(x_1) + bx_1x_2 + v$

Now, we let the part

$|4ax_1 \sin(x_1) + bx_1x_2| \leq 4\rho|x_1| + \sigma|x_1||x_2|$

Where $a \leq \rho$ and $b \leq \sigma$, and let's pick $\rho = 1$ *and* $\sigma = 2$. then,

$|4ax_1 \sin(x_1) + bx_1x_2| \leq \beta(x) = 4|x_1| + 2|x_1||x_2|$

Now, $\beta(x)$ will to be:

$\beta(x) = 4|x_1| + 2|x_1||x_2| + k$

where $k > 0$, so let's make it $k = 1$. Then, v will be:

$v = -\beta(x) * \mathrm{sgn}(s)]$

$v = -[4|x_1| + 2|x_1||x_2| + 1] * \mathrm{sgn}(s)]$

Suppose that v in the controller, we will get that

$u = -4x_2 + v$

$u = -4x_2 - [4|x_1| + 2|x_1||x_2| + 1] * \text{sgn}(s)$

We conclude that $(s = 0)$ is GUAS, and it can be proved that $s(t) \Longrightarrow 0$ in finite time.

Simulate the sliding mode design and the 'sign' function in Matlab, ode45 will take a very long time to run. Instead, it is suggested that we used the approximation $sgn(y) \approx sat(\frac{y}{\epsilon})$

Here, we will simulate the sliding mode design by $\text{sgn}(y) \approx sat(y/\varepsilon)$ function in MATLAB

Where $\boldsymbol{sat}(\boldsymbol{y}) = \begin{cases} \mathbf{1} & \textbf{if} \quad \boldsymbol{y} \geq \mathbf{1} \\ \boldsymbol{y} & \textbf{if} \ -\mathbf{1} < \boldsymbol{y} < \mathbf{1} \\ -\mathbf{1} & \textbf{if} \quad \boldsymbol{y} \leq -\mathbf{1} \end{cases}$, as tune ε to understand the trade-off involved

We use saturation function and let $\epsilon = 0.01$. The sliding mode controller using saturation function is:

$u = -4x_2 - [4|x_1| + 2|x_1||x_2| + 1] * \text{sat}(s)$

The simulation results will be shown in Figure (5) and (6) for the states of the plant and the phase portrait plot.

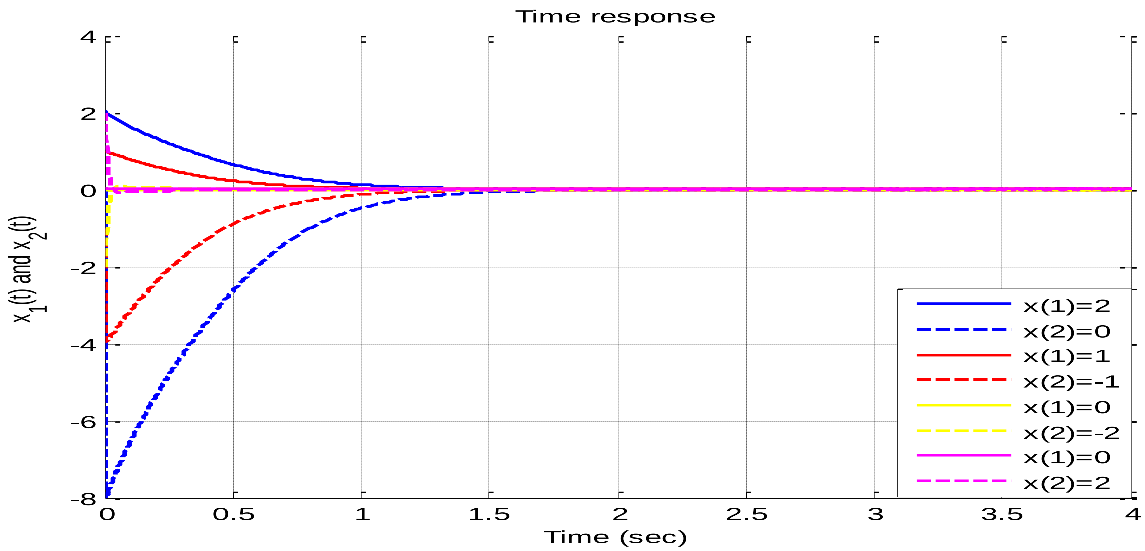

Figure (5): The states vector x of the plan

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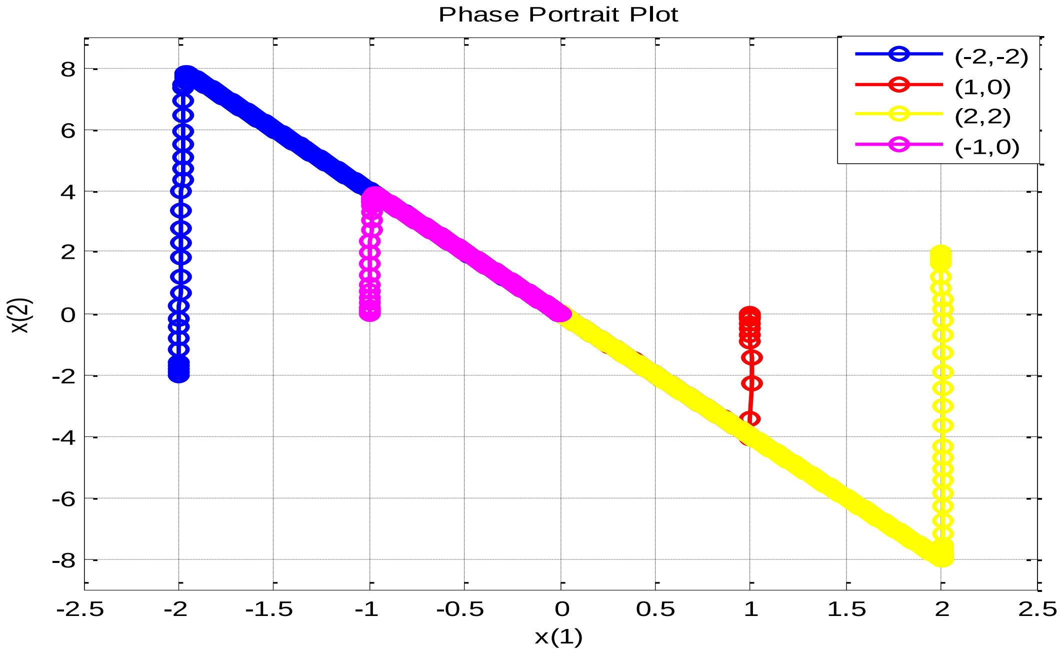


Figure (6): The phase portrait of the plant.

By using MATLAB we have found Figure (5) and (6) show the designed controller of the system regulates all states to zero.

3. Sliding Mode Control for the van der Pol Equation.

Sliding mode control is an area of increasing interest in control engineering , and this method is proved to be robust against disturbances and discrepancies between the physical plant and its mathematical model. However, it has mainly been applied to linear systems and its application to nonlinear systems is based on utilizing linear sliding surfaces and the controlled van der Pol system is given by:

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\omega^2 x_1 + \varepsilon\omega(1-\mu^2 x_1^2)x_2 u$

Where ω , ε and μ are positive constants and u is the control input.

We will pick some values for ω , ε and μ and verify via simulation that for $u = 1$ and the van der Pol system exhibits a stable limit cycle outside the surface $x_1^2 + x_2^2/\omega^2 = 1/\mu^2$ and that $u =$

There exists an unstable limit cycle outside the same surface, and we can find that limit cycle or oscillation is one of the most important phenomena that occur in dynamical systems and the system oscillates when it has a nontrivial periodic solution

$x(t+T)=x(t), \forall t \geq 0$ For some $T > \mathbf{0}$. Also, for $\boldsymbol{u} = \boldsymbol{1}$ this is the standard van der Pol oscillator which is known to have a stable limit cycle. And, the fact that the limit cycle is outside a circle of radius by $\mathbf{1/\mu}$ in the plane $(\boldsymbol{x_1}\ ,\ \boldsymbol{x_2/\omega})$ can be shown by transforming the Equation into polar coordinates.

The surface $x_1^2 + x_2^2/\omega^2 = 1/\mu^2$ is re-written as:

$$\frac{x_2^2}{\omega^2} + x_1^2 = \frac{1}{\mu^2}$$

We let the values $\omega = 1,\ \epsilon = 1$, and $\mu = 1$.

$$x_2^2 + x_1^2 = 1$$

when $u = 1$, the system will be:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2$$

Here we will shows the phase portrait plot when $u = 1$ in Figures (7)

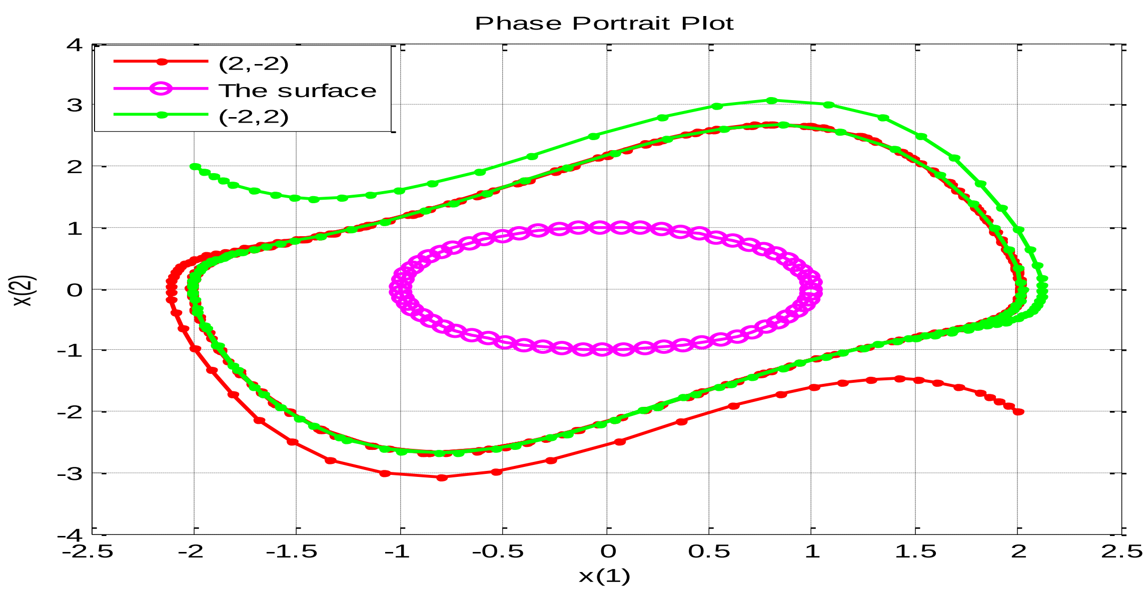


Figure (7): The phase portrait of the plant when $u = 1$

From Figure (7), the conclusion is that this limit cycle is stable outside the surface $\boldsymbol{x_1^2 + x_2^2/\omega^2 = 1/\mu^2}$ **.** Since that, all trajectories on this limit cycle must go to the outside. Therefore, the stable limit must be outside the circle.

When the control input $u = -1$, the system will be:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - (1 - x_1^2)x_2$$

The simulation result of the phase portrait plot when $u = -1$ is shown in Figure (8).

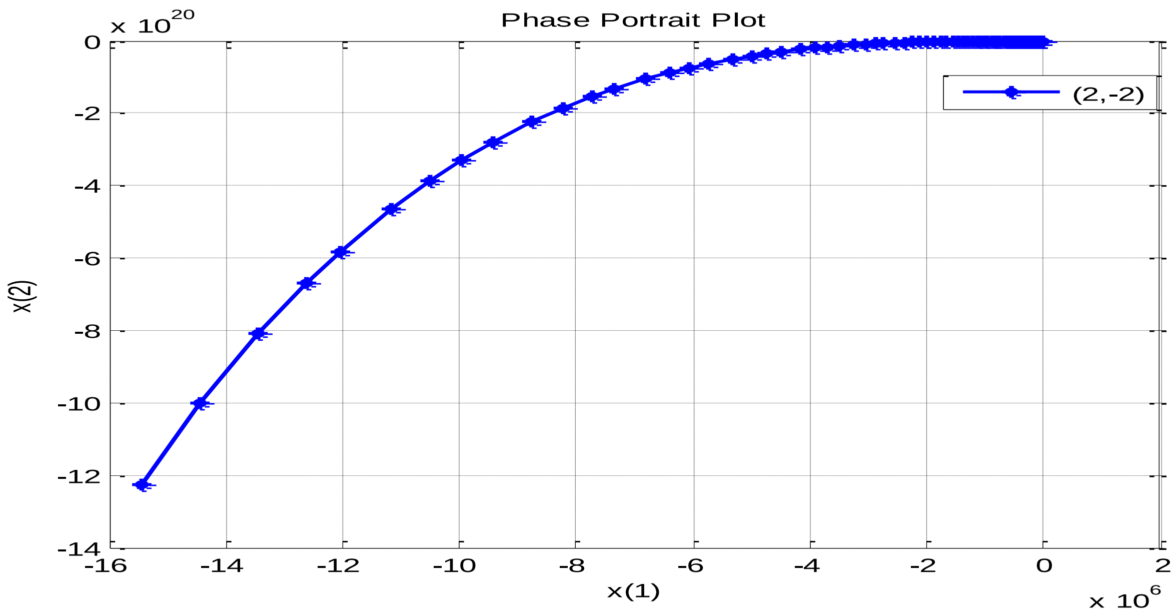


Figure (8): The phase portrait of the plant when $u = -1$

From Figure (8), we can conclude that the existence of the unstable limit cycle by reversing time and scaling the states, so it is unstable limit cycle outside the surface $\boldsymbol{x_1^2 + x_2^2/\omega^2 = 1/\mu^2}$**.**

Now we will define the sliding manifold $\boldsymbol{s = x_1^2 + x_2^2/\omega^2 - r^2}$, where $\boldsymbol{r < 1/\mu}$ and we will show that if restrict the motion of the system to the surface $\boldsymbol{s = 0}$, and that's we force $\boldsymbol{s(t) \equiv 0}$

Then the resulting behavior is that of the linear harmonic oscillator

$\boldsymbol{\dot{x}_1 = x_2}$ and $\boldsymbol{\dot{x}_2 = -\omega^2 x_1}$ which exhibits a sinusoidal oscillation of frequency $\boldsymbol{\omega}$ and amplitude $\boldsymbol{r}$.

We can write dynamics of the manifold:

$\dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{x}_{\boldsymbol{2}}$

$\dot{\boldsymbol{x}}_{\boldsymbol{2}} = -\boldsymbol{\omega}^{\boldsymbol{2}}\, \boldsymbol{x}_{\boldsymbol{1}} + \boldsymbol{\varepsilon\omega}(1-\boldsymbol{\mu}^{\boldsymbol{2}}\boldsymbol{x}_{\boldsymbol{1}}^{\boldsymbol{2}})\boldsymbol{x}_{\boldsymbol{2}}\mathrm{u}$

$s = x_1^2 + \frac{x_2^2}{\omega^2} - r^2 \implies \dot{s} = 2x_1\dot{x}_1 + 2\, {x_2\,\dot{x}_2}/{\omega^2} - r^2$

We let $r = 0.6$, and the sliding manifold is:

$s = x_1^2 + \frac{x_2^2}{\omega^2} - r^2 \quad \implies \quad s = x_1^2 + x_2^2 - (0.5)^2 = 0$

Hence, we can found the dynamics of the sliding manifold:

$$\dot{s} = \frac{\partial s}{\partial x}$$

$$\dot{s} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$\dot{s} = 2x_1x_2 + 2x_2(-x_1 + (1 - x_1^2)x_2\, u)$$

$$\dot{s} = 2x_1x_2 - 2x_1x_2 + 2(1 - x_1^2)x_2^2\, u$$

$$\dot{s} = 2(1 - x_1^2)x_2^2\, u$$

$\dot{s}(t) \equiv 0 \implies s(t) \equiv 0 \implies u(t) \equiv 0$ Then the resulting behavior the linear harmonic oscillator from the original Equation for system when the input zero $u(t) \equiv 0$ we can get the state equation reduces to a harmonic oscillator:

$\dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{x}_{\boldsymbol{2}}$ and $\dot{\boldsymbol{x}}_{\boldsymbol{2}} = -\boldsymbol{\omega}^{\boldsymbol{2}}\, \boldsymbol{x}_{\boldsymbol{1}} + \boldsymbol{\varepsilon\omega}(1-\boldsymbol{\mu}^{\boldsymbol{2}}\boldsymbol{x}_{\boldsymbol{1}}^{\boldsymbol{2}})\boldsymbol{x}_{\boldsymbol{2}}\mathrm{u} \implies \dot{\boldsymbol{x}}_{\boldsymbol{2}} = -\boldsymbol{\omega}^{\boldsymbol{2}}\boldsymbol{x}_{\boldsymbol{1}}$

In this part we will designing a sliding mode controller that driver or leads all trajectories whose initial is within the region $\{\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{2}}\!:\!|\boldsymbol{x}_{\boldsymbol{1}}|\ \boldsymbol{1/\mu}\,\}$ to the manifold $s = 0$ and then has the states slide on the manifold towards the origin. Also, we will simulate the controller and verify that it's able to regulate the state to zero.

$$V(s) = \frac{1}{2}s^2$$

$$\dot{V}(s) = s\dot{s} = s\; 2(1 - x_1^2)x_2^2\, u$$

When we make $s \neq 0$, we have to let the control input is $u = -\mathrm{sgn}(s)$. So, it will be:

$$s\dot{s} = 2(1 - x_1^2)x_2^2 * s * \text{sgn}(s)$$

$$s\dot{s} = 2(1 - x_1^2)x_2^2 * |s|$$

The simulation results with different initial conditions will be:

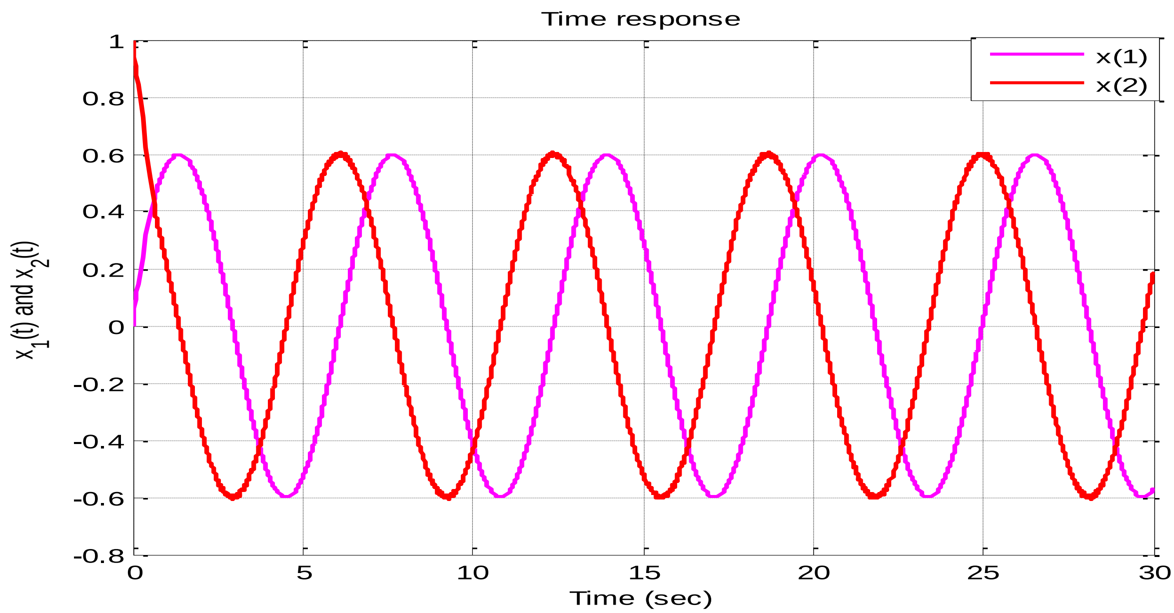


Figure (9): The states vector x of the plant.

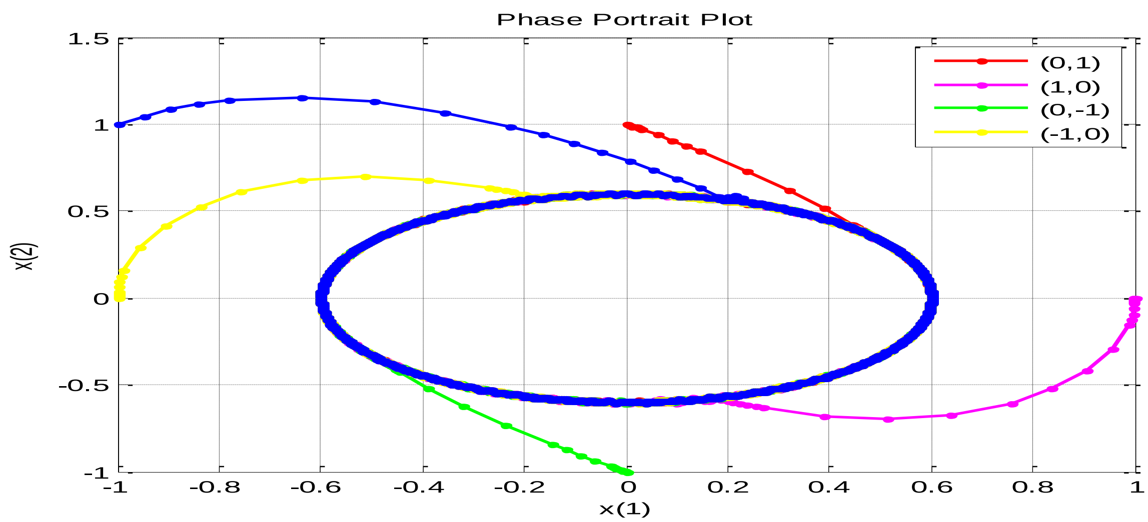


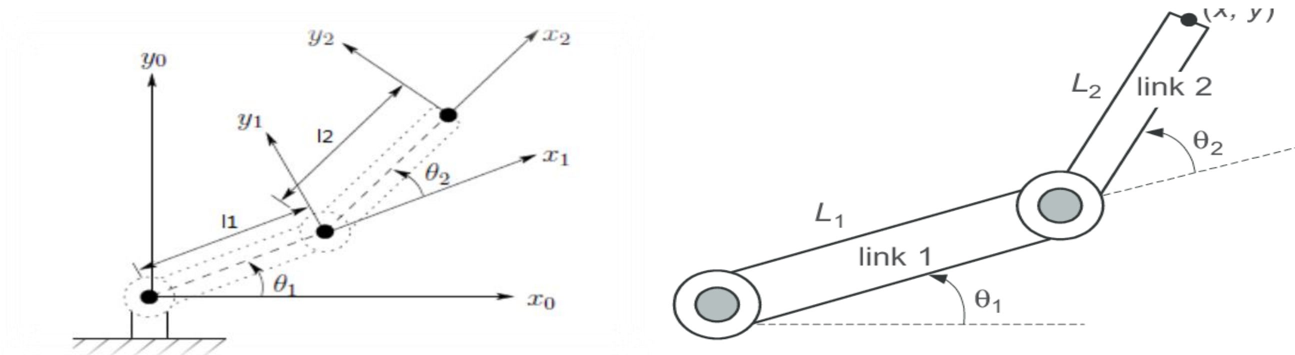
Figure (10): The phase portrait of the plant.

From the Figures (9) and (10), we can show that when $x_2 = 0$, there will be no trajectory shown. Also, we conclude that all trajectories in Figure (10) reach the sliding manifold $\boldsymbol{s = x_1^2 + \frac{x_2^2}{\omega^2} - r^2}$.

4. MIMO Control of a Two-Link Planar Arm.

The Multiple input, multiple output (MIMO) systems describe processes with more than one input and more than one output which require multiple control loops and Single variable input or single variable Output (SISO) control schemes are just one type of control scheme that engineers in industry use to control their process. In this part, we will consist of two links and the first one mounted on a rigid base by means of a frictionless hinge and the second mounted at the end of link one. The joint axes z_0 and z_1 and we establish the base frame $x_0 y_0 z_0$ as the work space frame, which means the arm moves within the $x-y$ plane. The inputs to the system are always the torques τ_1 and τ_2 applied at the joints.

Two - link planar manipulator.



A dynamic model of this system can be derived using Lagrangian equations and is given by

$$\begin{bmatrix} \boldsymbol{H_{11}} & \boldsymbol{H_{12}} \\ \boldsymbol{H_{21}} & \boldsymbol{H_{22}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}}_1 \\ \ddot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{h}\dot{\boldsymbol{\theta}}_2 & -\boldsymbol{h}\dot{\boldsymbol{\theta}}_1 - \boldsymbol{h}\dot{\boldsymbol{\theta}}_2 \\ \boldsymbol{h}\dot{\boldsymbol{\theta}}_1 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}}_1 \\ \dot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau_1} \\ \boldsymbol{\tau_2} \end{bmatrix}$$ where

$$H_{11} = I_1 + I_2 + m_1 l_{c1}^2 + m_2[l^2{}_1 + l^2{}_{c2} + 2 l_1 l_{c2} \cos(\theta_2)],$$

$$H_{22} = I_2 + m_2 l_{c2}^2,$$

$H_{12} = H_{21} = I_2 + m_2[l_{c2}{}^2 + l_1 l_{c2}\cos(\theta_2)],$

$h = m_2 l_1 l_{c2}\sin(\theta_2),$

$g_1 = m_1 l_{c1} g\cos(\theta_1) + m_2 g[l^2{}_{c2}\cos(\theta_1+\theta_2) + l_1\cos(\theta_1)],$

$g_2 = m_2 l_{c2} g\cos(\theta_1+\theta_2)$

In this part, we use the following parameter values $m_1 = 1.0kg$, mass of link one $m_2 = 1.0kg$, mass of link two $l_1 = 1.0m$ length of link one $l_2 = 1.0m$, length of link two $l_{c1} = 0.5m$, distance from the joint of link one to its center of gravity $l_{c2g} = 0.5m$, distance from the joint of link two to its center of gravity $I_1 = 0.2kgm^2$ lengthwise centroid inertia of link one $I_2 = 0.2kgm^2$, lengthwise centroidal inertia of link two and $g = 9.804m/s^2$, acceleration of gravity.

we will do joint space MIMO control of the arm, where the inputs are the two torques and the outputs are the joint angles θ_1 and θ_2 . Write a MIMO state space representation

of the system dynamics. Then design the MIMO state feedback controllers ($\boldsymbol{\tau_1}$ ***and*** $\boldsymbol{\tau_2}$) for

this system that regulate all the states to zero, and simulate the closed loop system for the initial conditions $\boldsymbol{\theta_1(0)} = \frac{\pi}{2}$, $\dot{\theta}_1(0) = 0$, $\boldsymbol{\theta_2(0)} = -\frac{\pi}{2}$, $\dot{\theta}_2(0) = 0$,we will to plot the controllers and all the states versus time, and provide an interpretation of the plot.

$$\begin{bmatrix} \boldsymbol{H_{11}} & \boldsymbol{H_{12}} \\ \boldsymbol{H_{21}} & \boldsymbol{H_{22}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}}_1 \\ \ddot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{h}\dot{\boldsymbol{\theta}}_2 & -\boldsymbol{h}\dot{\boldsymbol{\theta}}_1 - \boldsymbol{h}\dot{\boldsymbol{\theta}}_2 \\ \boldsymbol{h}\dot{\boldsymbol{\theta}}_1 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}}_1 \\ \dot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau_1} \\ \boldsymbol{\tau_2} \end{bmatrix} \quad \text{where}$$

$H_{11} = I_1 + I_2 + m_1 l_{c1}{}^2 + m_2[l^2{}_1 + l^2{}_{c2} + 2l_1 l_{c2}\cos(\theta_2)],$

$H_{22} = I_2 + m_2 l_{c2}{}^2,$

$H_{12} = H_{21} = I_2 + m_2[l_{c2}{}^2 + l_1 l_{c2}\cos(\theta_2)],$

$h = m_2 l_1 l_{c2}\sin(\theta_2),$

$g_1 = m_1 l_{c1} g\cos(\theta_1) + m_2 g[l^2{}_{c2}\cos(\theta_1+\theta_2) + l_1\cos(\theta_1)],$

$g_2 = m_2 l_{c2} g\cos(\theta_1+\theta_2)$

$H_{11}\ddot{\theta} + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2\dot{\theta}_1 - h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 + g_1 = \tau_1$

$$H_{21}\ddot{\theta} + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 + g_2 = \tau_2$$

$$\ddot{\theta}_1 = -\frac{H_{12}}{H_{11}}\ddot{\theta}_2 + \frac{h}{H_{11}}\dot{\theta}_2\dot{\theta}_1 + \frac{h}{H_{11}}\dot{\theta}_1\dot{\theta}_2 + \frac{h}{H_{11}}\dot{\theta}_2^2 - \frac{g_1}{H_{11}} + \frac{\tau_1}{H_{11}}$$

$$\ddot{\theta}_2 = -\frac{H_{21}}{H_{22}}\ddot{\theta_1} - \frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g}{H_{22}} + \frac{\tau_2}{H_{22}}$$

$$\ddot{\theta}_1 = -\frac{H_{21}}{H_{22}}\left[-\frac{H_{21}}{H_{22}}\ddot{\theta}_1 - \frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}} + \frac{\tau_2}{H_{22}}\right] + \frac{h}{H_{11}}\dot{\theta}_2\dot{\theta}_1 + \frac{h}{H_{11}}\dot{\theta}_1\dot{\theta}_2 + \frac{h}{H_{11}}\dot{\theta}_2^2 - \frac{g_1}{H_{11}} + \frac{\tau_1}{H_{11}}$$

We will define that:

$$M = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1} \begin{bmatrix} -h\dot{\theta}_2 & -h\dot{\theta}_1 - h\dot{\theta}_2 \\ h\dot{\theta}_1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1} \begin{bmatrix} g1 \\ g2 \end{bmatrix}$$

$$Hinv = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1}$$

And define that:

$$q_1 = \theta_1$$
$$q_2 = \theta_2$$
$$q_3 = \dot{\theta}_1$$
$$q_4 = \dot{\theta}_2$$

Then the state space representation of the system will be:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & M(1,1) & M(1,2) \\ 0 & 0 & M(2,1) & M(2,2) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G(1,1) \\ G(2,1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Hinv(1,1) & Hinv(1,2) \\ Hinv(2,1) & Hinv(2,2) \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Then we can start to design our control torque input. Note that we have to control q1 and q2 independently, so that we have decoupled the 2 inputs.

Than we will define that:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \tau = Hu = H \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$ where, *Hinv*H=I*

And let that:

$$u_1 = -M(1,1)q_3 - M(1,2)q_4 - G(1,1) - k_{11}\, q_1 - k_{12}q_3$$
$$u_2 = -M(2,1)q_3 - M(2,2)q_4 - G(2,1) - k_{21}\, q_2 - k_{22}q_4$$

Where, H_{11}=H_{12}=H_{22}=H_{22} =1

$$\ddot{\theta}_1 = \frac{H_{12}^2\ddot{\theta}_1}{H_{11}H_{22}} + \frac{hH_{12}}{H_{22}H_{11}}\dot{\theta}_1^2 + \frac{g_2H_{12}}{H_{11}H_{12}} - \frac{\tau_2H_{12}}{H_{11}H_{12}} + \frac{h}{H_{11}}\dot{\theta}_2\dot{\theta}_1 + \frac{h}{H_{11}}\dot{\theta}_1\dot{\theta}_2 + \frac{h}{H_{11}} + \frac{h}{H_{11}}\dot{\theta}_2^2 - \frac{g_1}{H_{11}} + \frac{\tau_1}{H_{11}}$$

$$\ddot{\theta}_1 = \frac{H_{11}H_{22}}{H_{11}H_{22}-H_{12}^2}\left[\frac{hH_{12}}{H_{22}H_{11}}\dot{\theta}_1^2 + \frac{g_2H_{12}}{H_{11}H_{22}} - \frac{H_{12}}{H_{11}H_{22}}\tau_2 + 2\frac{h}{H_{11}}\dot{\theta}_2\dot{\theta}_1 + + \frac{h}{H_{11}}\dot{\theta}_2^2 - \frac{g_1}{H_{11}} + \frac{\tau_1}{H_{11}}\right]$$

$$\ddot{\theta}_1 = \frac{1}{H_{11}H_{22}-H_{12}^2}\left[h\dot{\theta}_1^2H_{12} + g_2H_{12} - H_{12}\tau_2 + 2h\dot{\theta}_1\dot{\theta}_2H_{12} + h\dot{\theta}_2^2H_{12} - g_1H_{12} + H_{12}\tau_1\right]$$

$$\ddot{\theta}_1 = \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\left(h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2 + \tau_1 - \tau_2\right)$$

$$\ddot{\theta}_2 = -\frac{H_{21}}{H_{22}}\left[\frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\left(h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2 + \tau_1 - \tau_2\right)\right] - \frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}} + \frac{\tau_2}{H_{22}}$$

$$\ddot{\theta}_2 = -\frac{H_{12}^2}{H_{22}[H_{11}H_{22}-H_{12}^2]}\left[h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2 + \tau_1 - \tau_2\right] - \frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}} + \frac{\tau_2}{H_{22}} \quad \dot{\theta}_1 =$$
θ_1

$$\ddot{\theta}_1 = \frac{H_{11}H_{22}}{H_{11}H_{22}-H_{12}^2}\left[h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2 + \tau_1 - \tau_2\right]$$

$$\dot{\theta}_2 = \theta_2$$

$$\ddot{\theta}_2 = -\frac{H_{12}^2}{[H_{11}H_{22}-H_{12}^2]}\left[\frac{h\dot{\theta}_1^2}{H_{22}} + \frac{h\dot{\theta}_2^2}{H_{22}} - \frac{g_1}{H_{22}} + \frac{g_2}{H_{22}} + \frac{2h\dot{\theta}_1\dot{\theta}_2}{H_{22}} + \frac{\tau_1}{H_{22}} - \frac{\tau_2}{H_{22}}\right] - \frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}} + \frac{\tau_2}{H_{22}}$$

$$x_1 = \theta_1 \quad , \quad x_2 = \dot{\theta}_1 \quad , \quad x_3 = \theta_2 \quad , \quad x_4 = \dot{\theta}_2$$

$$\dot{x}_1 = x_2$$

$$x_2 = \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\left[hx_2^2 + hx_4^2 - g_1 + g_2 + 2hx_2x_4 + \tau_1 - \tau_2\right]$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{H_{12}^2}{[H_{11}H_{22}-H_{12}^2]}\left[\frac{hx_2^2}{H_{22}} + \frac{h\dot{x}_4^2}{H_{22}} - \frac{g_1}{H_{22}} + \frac{g_2}{H_{22}} + \frac{2hx_2x_4}{H_{22}} + \frac{\tau_1}{H_{22}} - \frac{\tau_2}{H_{22}}\right] - \frac{h}{H_{22}}x_2^2 - \frac{g_2}{H_{22}} + \frac{\tau_2}{H_{22}}$$

$$g_1 = \theta_1 \quad , \quad \dot{g}_1 = \dot{\theta}_1$$

$$\ddot{g}_1 = \ddot{\theta}_1 = \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\left(h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2 + \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\tau_1 - \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\tau_2\right)$$

$g_2 = \theta_2$ and $\dot{g}_2 = \dot{\theta}_2$

$$\ddot{g}_2 = \ddot{\theta}_2 = -\frac{H_{12}^2}{H_{11}H_{22}-H_{12}^2}\left[\frac{h\dot{\theta}_1^2}{H_{22}} + \frac{h\dot{\theta}_1^2}{H_{22}} - \frac{g_1}{H_{22}} + \frac{g_2}{H_{22}} + \frac{2h\theta_1\dddot{\theta}_2}{H_{22}}\right]\frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}} + \frac{1}{H_{22}}\left[-\frac{H_{12}^2}{H_{11}H_{22}-H_{12}^2} + 1\right]\tau_1 + \left[\frac{H_{12}^2}{H_{22}[H_{11}H_{22}-H_{12}^2]}\right]\tau_2$$

$$F_1(x) = \frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\left(h\dot{\theta}_1^2 + h\dot{\theta}_2^2 - g_1 + g_2 + 2h\dot{\theta}_1\dot{\theta}_2\right)$$

$$F_2(x) = -\frac{H_{12}^2}{H_{11}H_{22}-H_{12}^2}\left[\frac{h\dot{\theta}_1^2}{H_{22}} + \frac{h\dot{\theta}_1^2}{H_{22}} - \frac{g_1}{H_{22}} + \frac{g_2}{H_{22}} + \frac{2h\theta_1\dddot{\theta}_2}{H_{22}}\right]\frac{h}{H_{22}}\dot{\theta}_1^2 - \frac{g_2}{H_{22}}$$

$$g_1(x) = \left[\frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\tau_1\right] \quad , \quad g_2(x) = \left[-\frac{H_{12}}{H_{11}H_{22}-H_{12}^2}\tau_2\right]$$

$$g_3(x) = \left[\left(\frac{1}{H_{22}} - \frac{H_{12}}{H_{11}H_{22}-H_{12}^2} + 1\right)\tau_1\right] \quad , \quad g_4(x) = \left[\frac{H_{12}^2}{H_{22}[H_{11}H_{22}-H_{12}^2]}\tau_2\right]$$

$$\ddot{g}_1 = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} + \begin{bmatrix} g_1(x) & g_2(x) \\ g_3(x) & g_4(x) \end{bmatrix}\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} g_1(x) & g_2(x) \\ g_3(x) & g_4(x) \end{bmatrix} - \left(\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} + v\right)$$

$$v_1 = -k_1 g_1 - k_2\dot{g}_1 = 2x_1 - 3x_2$$

$$v2 = -k_1 g_2 - k_2\dot{g}_2 = 24x_3 - 10x_4$$

$$u_1 = -\frac{g_2}{g_1}\left[\frac{1}{g_1 g_4 - g_2 g_3}(g_3 f_1 - g_3 v_1 - g_1 f_2 + g_1 - \frac{f_1}{g_1} + \frac{v_1}{g_1}\right]$$

$$u_1 = \left[\frac{1}{g_1 g_4 - g_3 g_2}(g_3 f_1 - g_3 v_1 - g_1 f_1 + g_1 v_2\right]$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \frac{1}{H_{11}H_{22}-H^2}\begin{bmatrix} H_{22} & -H_{21} \\ H_{21} & H_{11} \end{bmatrix}\begin{bmatrix} 2h\dot{\theta}_1\dot{\theta}_2 + h\dot{\theta}_2^2 - g_1 + \tau_1 \\ -h\dot{\theta}_1^2 - g_2 + \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \frac{1}{H_{11}H_{22}-H^2}\begin{bmatrix} H_{22}\left(2h\dot{\theta}_1\dot{\theta}_2 + h\dot{\theta}_2^2 - g_1 + \tau_1\right) - H_{21}(-h\dot{\theta}_1^2 - g_2 + \tau_2) \\ (\; -H_{21}\left(2h\dot{\theta}_1\dot{\theta}_2 + h\dot{\theta}_2^2 - g_1 + \tau_1\right) + H_{11}\,(h\dot{\theta}_1^2 - g_2 + \tau_2) \end{bmatrix}$$

$$\ddot{\theta}_1 = \frac{H_{22}}{H_{11}H_{22}-H^2}\left(2h\dot{\theta}_1\dot{\theta}_2 + h\dot{\theta}_2^2 - g_1 + \tau_1\right) - \frac{H_{11}}{H_{11}H_{22}-H^2}\left(-h\dot{\theta}_1^2 - g_2 + \tau_2\right)$$

$$\dot{x}_2 = \frac{H_{22}}{H_{11}H_{22}-H^2}\left(2hx_2x_4 + hx_4^2 - g_1 + u_1\right) - \frac{H_{21}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right)$$

$$\ddot{\theta}_2 = \frac{-H_{21}}{H_{11}H_{22}-H^2}\left(2h\dot{\theta}_1\dot{\theta}_2 + h\dot{\theta}_2^2 - g_1 + \tau_1\right) - \frac{H_{11}}{H_{11}H_{22}-H^2}\left(-h\dot{\theta}_1^2 - g_2 + \tau_2\right)$$

$$\dot{x}_4 = \frac{-H_{21}}{H_{11}H_{22}-H^2}\left(2hx_2x_4 + hx_4^2 - g_1 + u_1\right) - \frac{H_{21}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right)$$

$$y_1 = x_1 \quad \Rrightarrow \quad y_2 = x_2$$

$$H_{11}H_{22} - H^2 = 0.405 - 1.1125\cos\theta_2 - 0.25\cos\theta_2^2$$

$$H_{11}H_{22} - H^2 = 0.405 - 1.1125\cos x_3 - 0.25\cos x_1^2$$

$$x_1 = \theta_1 \quad , \quad x_2 = \dot{\theta}_1 \quad , \quad x_3 = \theta_2 \quad , x_4 = \dot{\theta}_2$$

$$\dot{y}_1 = \dot{x}_1 = x_2 \quad \Rrightarrow \quad \ddot{y}_1 = \dot{x}_2$$

$$\dot{y}_2 = \dot{x}_3 = x_4 \quad \Rrightarrow \quad \ddot{y}_2 = \dot{x}_4$$

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} = \begin{bmatrix}\frac{H_{22}}{H_{11}H_{22}-H^2}(2hx_2x_4 + hx_4^2 - g_1) - \frac{H_{21}}{H_{11}H_{22}-H^2}(-hx_2^2 - g_2)\\ \frac{-H_{21}}{H_{11}H_{22}-H^2}(2hx_2x_4 + hx_4^2 - g_1) + \frac{H_{11}}{H_{11}H_{22}-H^2}(-hx_2^2 - g_2)\end{bmatrix} +$$

$$\begin{bmatrix}\frac{H_{22}}{H_{11}H_{22}-H^2} & -\frac{H_{21}}{H_{11}H_{22}-H^2}\\ -\frac{H_{21}}{H_{11}H_{22}-H^2} & \frac{H_{11}}{H_{11}H_{22}-H^2}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}\frac{1}{H_{11}H_{22}-H^2}(2hH_{22}x_2x_4 + H_{22}hx_4^2 - H_{22}g_1 + H_{21}hx_2^2 + H_{21}g_2)\\ \frac{1}{H_{11}H_{22}-H^2}(-2hH_{22}x_2x_4 - H_{22}hx_4^2 + H_{22}g_1 + H_{21}hx_2^2 + H_{21}g_2)\end{bmatrix} -$$

$$\begin{bmatrix}K_{11} & K_{12} & 0 & 0\\ 0 & 0 & K_{23} & K_{24}\end{bmatrix}\begin{bmatrix}y_1\\ \dot{y}_1\\ y_2\\ \dot{y}_2\end{bmatrix}$$

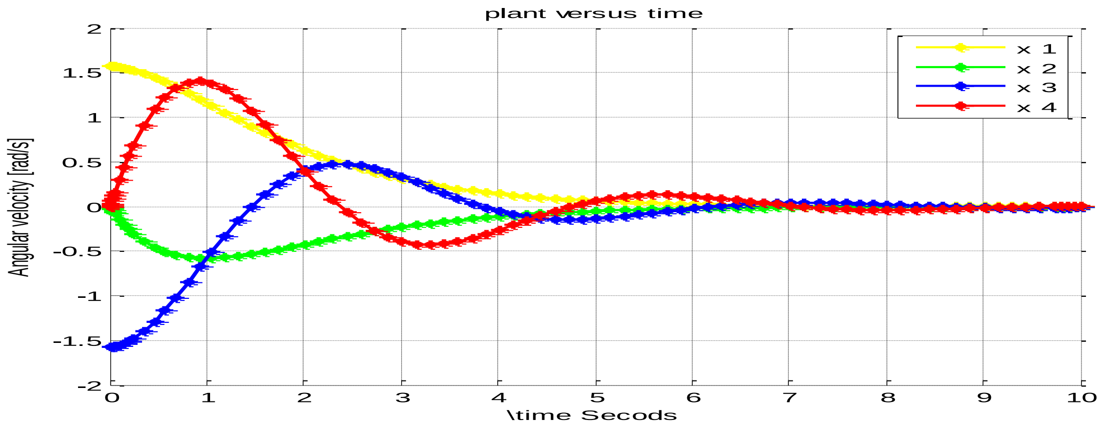


Figure (11): Angular velocity [rad/s]

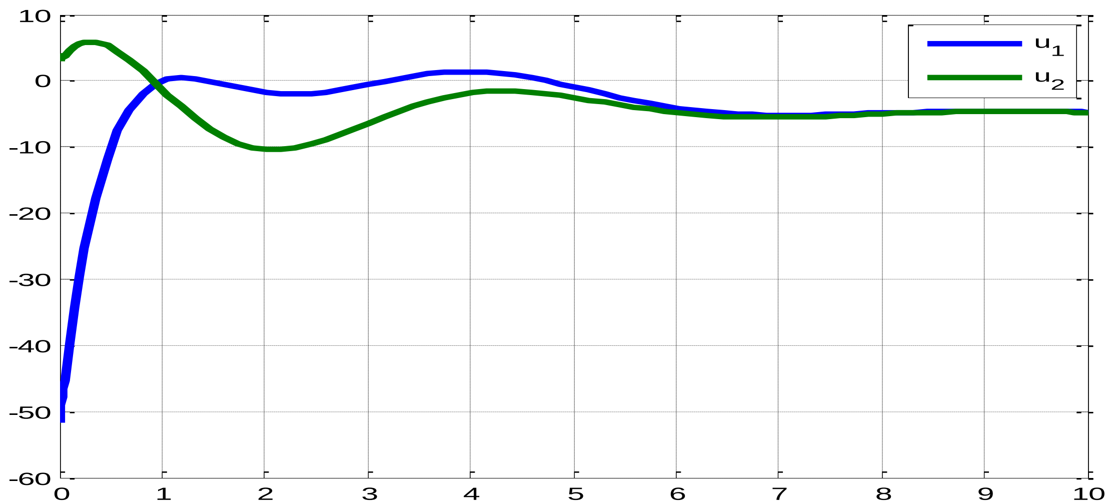


Figure (12): input u_1 and u_2

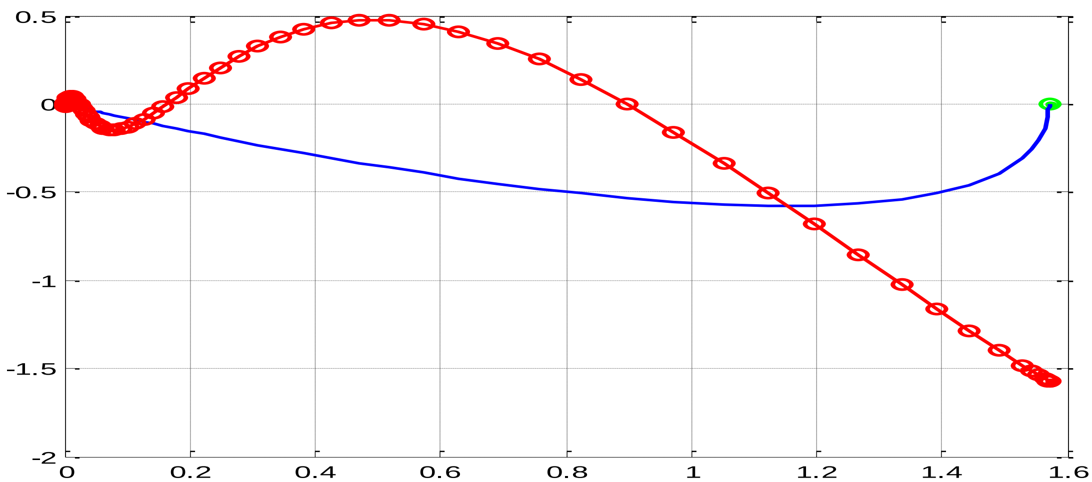


Figure (13): Plot x (1,1), x(1,2) and y1(1), y2(1)

Now we want to control the end-effector of the arm to generate a path in $x-y$ plan. The output s are the position coordinates x and y (z is always zero) of the end-effector. Let

Be a smooth and invertible mapping between the joint vector $\theta=[\theta_1,\theta_2]^T \in Q$ and the workspace variables. The space Q is a suitably chosen rage of angles for θ_1 and θ_2, then $X=[X_0,Y_0]^T$ is the position of the end-effector on the (x_0-y_0) plane. In other words $X=f(\theta)$, so that the position of the end-effector can be computed from the two joint angles, and mapping f is known as the arm's forward kinematics, and it can lead us directly into so-called work space control, which means that we will perform instead of joint space control. Also, the exists a relationship between the vector of linear velocities of the end-effector $\dot{\boldsymbol{X}}$ and the vector of velocities of joint variables $\dot{\boldsymbol{\theta}}$ given by $\dot{\boldsymbol{X}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$ where $\boldsymbol{J}$ is as the Jacobian matrix, and the forward kinematic function , and we will using trigonometry to show that the Jacobian is given by:

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

Then, the derive the new MIMO state space representation of the system dynamics, where the state vector is X instead of Θ. We will design the MIMO state feedback controllers ($\hat{\tau}_1$ and $\hat{\tau}_2$)

$$\begin{cases} \boldsymbol{x} = \sqrt{2} \ (\boldsymbol{m}) \\ \boldsymbol{y} = \boldsymbol{sin}\left(\frac{\boldsymbol{\pi t}}{\boldsymbol{30}}\right) (\boldsymbol{m}) \end{cases}$$

And we should verify design with simulation for initial conditions$\boldsymbol{\theta_1} = (\boldsymbol{0})$, $\dot{\boldsymbol{\theta}}_1 = (\boldsymbol{0})$, $\dot{\boldsymbol{\theta}}_2 = \boldsymbol{0.2}$, $\dot{\boldsymbol{\theta}}_2 = (\boldsymbol{0})$. We will plot the control signals and all the states versus time, and demonstrate an animation of the arm's motion.

Here, we want to control the end-effector of the arm to generate a path in $x-y$ plan. The output s are the position coordinates x and y (z is always zero) of the end-effector.

$$\boldsymbol{x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)}$$

$$\boldsymbol{y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)}$$

Let's the position equation to find the velocity:

$$\boldsymbol{\dot{x} = l_1 \dot{\theta}_1 \sin\theta_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)}$$

$$\boldsymbol{\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2)}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} l_1 \sin\theta_1 - l_2 \sin\ (\theta_1+\theta_2) & -l_2(\theta_1+\theta_2) \\ l_1 \cos\theta_1 - l_2 \cos\ (\theta_1+\theta_2) & l_2(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$X \dot{=} J\Theta \quad \Rightarrow \quad \Theta = \gamma^{-1} X$$

$$y_1 = l_1 \cos x_1 + l_2 \cos(x_1 + x_3)$$

$$\ddot{y}_1 = -l_1 \dot{x}_1 \sin x_1 - l_2\ (\dot{x}_1 + \dot{x}_3) \sin(x_1 + x_3) \Rightarrow -l_1 x_2 \sin x_1 - l_2(x_2 + x_4) \sin(x_1 + x_3)$$

$$\ddot{y}_1 = -l_1 x_2 \sin x_1 - l_2 x_2 sin(x_1 + x_3) - l_2 x_4 \sin(x_1 + x_3)$$

$\ddot{y} = \quad -l_1(x_2\dot{x}_1\cos x_1 + \dot{x}_2\sin x_1) - l_2[x_2(\dot{x}_1+\dot{x}_3)cos(x_1+x_3) + \dot{x}_2 sin(x_1+x_3)] - l_2[x_4(\dot{x}_1{+}\dot{x}_3)\cos(x_1+x_3) + \dot{x}_4\sin(x_1{+}x_3)]$

$\ddot{y} = \quad -l_1x_2^2(\cos x_1 - l_1\dot{x}_2\sin x_1) - l_2x_2^2\cos(x_1+x_3) - l_2x_2x_4\cos(x_1+x_3) - l_2\dot{x}_2\sin(x_1+x_3) - l_2\,x_2x_4\cos(x_1+x_3) - l_2x_2^4\cos(x_1+x_3) - l_2\dot{x}_4\sin(x_1+x_3)$

$\ddot{y} = -l_1x_2^2\cos x_1 - \dot{x}_2[-l_1\sin x_1 - l_2\sin(x_{1+}x_3)] \quad - l_2x_2^4\cos(x_1+x_3) - l_2\dot{x}_4\sin(x_1+x_3) - 2l_2x_2x_4\cos(x_1+x_3) - l_2x_4^2\cos(x_1+x_3) - l_2x_4^2\sin(x_1+x_3)$

$\ddot{y}_1 = -l_1\sin x_1\left[\frac{H_{22}}{H_{11}H_{22}-H^2}(2hx_2x_4 + hx_4^2 - g_1 + u_1)\right] - \frac{H_{21}}{H_{11}H_{22}-H^2}(-hx_2^2 - g_2 + u_2) - l_2\sin(x_1+x_3)\left[\frac{H_{22}}{H_{11}H_{22}-H^2}(2hx_2x_4 + hx_4^2 - g_1 + u_1)\right] - \frac{H_{21}}{H_{11}H_{22}-H^2}(-hx_2^2 - g_2 + u_2) - l_2\sin(x_1+x_3)\left[-\frac{H_{21}}{H_{11}H_{22}-H^2}(2hx_2x_4 + hx_4^2 - g_1 + u_1)\right] + \frac{H_{11}}{H_{11}H_{22}-H^2}(-hx_2^2 - g_2 + u_2) - -l_1x_2^2\cos x_1 - -l_1x_2^2\cos(x_1+x_3) - 2l_2x_2x_4\cos(x_1+x_3)$

$\left[\frac{-l_1\sin x_1 H_{22}}{H_{11}H_{22}-H^2} - \frac{-l_2\sin(x_1+x_3)H_{22}}{H_{11}H_{22}-H^2} + \frac{l_2\sin(x_1+x_3)H_{22}}{H_{11}H_{22}-H^2}\right]u_1 \quad \Rightarrow \quad$ *The first equation* $\quad \Rightarrow \quad \ddot{\boldsymbol{y}}_{\mathbf{1}}$

$\left[\frac{l_1\sin x_1 H_{21}}{H_{11}H_{22}-H^2} + \frac{l_2\sin(x_1+x_3)H_{21}}{H_{11}H_{22}-H^2} - \frac{l_2\sin(x_1+x_3)H_{11}}{H_{11}H_{22}-H^2}\right]u_2 \quad \Rightarrow \quad$ *The second equation* $\quad \Rightarrow \quad \ddot{\boldsymbol{y}}_{\mathbf{1}}$

$y = l_1\sin x_1 + l_2\sin(x_1+x_3)$

$\dot{y} = l_1\dot{x}_1\cos x_1 + l_2\,(\dot{x}_1+\dot{x}_3)\cos(x_1+x_3)$

$\dot{y} = l_1\,x_2\cos x_1 + l_2x_2\cos(x_{1+}x_3) + l_2\,x_4\cos(x_{1+}x_3)$

$$\dot{y} = l_1[-x_2\dot{x}_1\sin x_1 + \dot{x}_2\cos x_1] + l_2[-x_2\sin(x_1+x_3)(\dot{x}_1+\dot{x}_3) + \dot{x}_2\cos(x_1+x_3)] + l_2[-x_4\sin(x_1+x_3)(\dot{x}_1+\dot{x}_3) + \dot{x}_4\cos(x_1+x_3)]$$

$\dot{y} = l_1\,[-x_2^2\sin x_1 + \dot{x}_2\cos x_1] + l_2[-x_2\sin(x_1+x_3)(x_2+x_4) + \dot{x}_2\cos(x_1+x_3)] + l_2[-x_4\sin(x_1+x_3)(x_2+x_4) + \dot{x}_4\cos(x_1+x_3)]$

$\dot{y} = -l_1\,x_2^2\sin x_1 + l_1\dot{x}_2\cos x_1 - l_2\sin(x_1+x_3)(x_2+x_4) + l_2\dot{x}_2\cos(x_1+x_3) - l_2x_4\sin(x_1+x_3)(x_2+x_4) + l_2\dot{x}_4\cos(x_1+x_3)$

$y\dot{} = \dot{x}_2[l_1\cos(x_1) + l_2\cos(x_1+x_3) + l_2\dot{x}_4\cos(x_1+x_3)]$

$\ddot{y}_2 = -l_1\,x_2^2\sin x_1 + l_1\dot{x}_2\cos x_1 - l_2\sin(x_1+x_3)(x_2+x_4) + l_2\dot{x}_2\cos(x_1+x_3) - l_2x_4\sin(x_1+x_3)(x_2+x_4) + l_2\dot{x}_4\cos(x_1+x_3)$

$\ddot{y}_2 = -l_1\, x_2^2 \sin x_1 + l_1\dot{x}_2 \cos x_1 - l_2\, x_2^2 \sin(x_1+x_3) - l_2 x_4 \sin(x_1+x_3) + l_2\dot{x}_2 \cos(x_1+x_3) - l_2 x_4 \sin(x_1+x_3) - l_2\, x_4^2 \sin(x_1+x_3) + l_2\dot{x}_4 \cos(x_1+x_3)$

$$\ddot{y}_2 = -l_1\, x_2^2 \sin x_1 + l_1\dot{x}_2 \cos x_1 \left[\frac{H_{22}}{H_{11}H_{22}-H^2}\left(2hx_2x_4 + hx_4^2 - g_1 + u_1\right) - \frac{H_{21}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right)\right] - l_2\, x_2^2 \sin(x_1+x_3) - l_2 x_4 \sin(x_1+x_3) + l_2 \cos(x_1+x_3)\left[\frac{H_{22}}{H_{11}H_{22}-H^2}\left(2hx_2x_4 + hx_4^2 - g_1 + u_1\right) - \frac{H_{21}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right)\right] - l_2\, x_2 x_4 \sin(x_1+x_3) - l_2\, x_2^2 \sin(x_1+x_3) + l_2 \cos(x_1+x_3)\left[-\frac{H_{21}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right) + \frac{H_{11}}{H_{11}H_{22}-H^2}\left(-hx_2^2 - g_2 + u_2\right)\right]$$

$$\left[\frac{l_1 \cos x_1 H_{22}}{H_{11}H_{22}-H^2} + \frac{l_2 \cos(x_1+x_3) H_{22}}{H_{11}H_{22}-H^2} - \frac{l_2 \sin(x_1+x_3) H_{21}}{H_{11}H_{22}-H^2}\right] u_1 \quad \Rightarrow \quad \textit{The first equation} \;\Rightarrow\; \ddot{\boldsymbol{y}}_{\boldsymbol{2}}$$

$$\left[-\frac{l_1 \cos x_1 H_{21}}{H_{11}H_{22}-H^2} - \frac{l_2 \cos(x_1+x_3) H_{21}}{H_{11}H_{22}-H^2} + \frac{l_2 \cos(x_1+x_3) H_{11}}{H_{11}H_{22}-H^2}\right] u_2 \quad \Rightarrow \quad \textit{The second equation} \;\Rightarrow\; \ddot{\boldsymbol{y}}_{\boldsymbol{2}}$$

5. Conclusions:

According to the results we got from this paper we found that the feedback linearization technique controls the system only within the certain constrains. Also, we have found the sliding manifold and we restrict the motion of the system to the surface $s = 0$. And, we have simulated the controller to regulate the state to zero.in third part, which means MIMO control of a Two-Link Planar Arm. In the analysis, nonlinear closed-loop system is assumed to have been designed and it is necessary to determine the characteristic of the system's behavior. In the design it is given a nonlinear plant to be controlled and some specifications of closed-loop system meets the desired characteristics. When a linear controller is used to control robot motion, it neglects the inherent nonlinear forces associated with the motion of the robot links. The

Controller's accuracy thus quickly degrades as the speed of motion increases, because many of the dynamic forces, such as Coriolis, centripetal forces, vary as square of the speed. However, in control systems there are much nonlinearity whose discontinuous nature does not allow linear approximation (friction, saturation, dead-zone, hysteresis and backlash).

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