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Solve the theory and design of a linear system to achieve stability and study the inverted pendulum on a cart using (MATLAB)

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الخلاصة

تتناول هذه الورقة البحثية المبادئ الأساسية لدراسة حل مشكلتين هما حل مسألة "نظرية وتصميم النظام الخطي" وضبط المرجع (r) ليكون سلسلة من الخطوات بالتناوب خلال فترة زمنية كافية لتحقيق حالة الاستقرار. أما المسألة الثانية فتتمثل في دراسة البندول المقلوب على عربة واختيار قيم مناسبة لمعاملات النظام باستخدام MATLAB لمحاكاة النظام باستخدام المعادلات غير الخطية الأصلية "نظام الحلقة المفتوحة".

Abstract:

This research paper addresses the basic principles of studying the solution of two problems: solving the problem of "linear system theory and design" and setting the reference (r) to be a series of alternating steps with a sufficient time interval to achieve a steady state, and the second problem is studying an inverted pendulum on a cart and choosing appropriate values for the system parameters. MATLAB is used to simulate the system using the original nonlinear equations (Open-loop system).

Key words: *linear system theory and design, the feed forward gain* (ρ), *state feedback gain (k), Reference (r), Open loop system, Feedback, The conrollability matrix , Pendulum on a cart*

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1. Introduction

The basic principles of this paper about two problems. The first problem is to solve problem “Linear System theory and Design” and set the reference (r) to be train of steps alternating between -4 and 4 with a period is chosen long enough that the steady state can take place (Neishtadt, 1982). Moreover, to show what happens when the selected period is decreased (Sussmann and Liu et al., 1994). The system is simulated with a feedback controller, and the plot reference and output are plotted in the same plot to demonstrate tracking (Arnol'd, V.I, 1989).

The second problem is to consider the inverted pendulum on a cart and choose reasonable values for the system’s parameters and using MATLAB (Bailliieul and Lehmam, 1996) to simulate the system using the original nonlinear equations “Open loop system” and then using the linearized model of the system to design a state-feedback controller that balances the pendulum and regulates the cart’s position to zero (Bogoliubov and Mitropolsky et al., 1961).

2. Background

We have the continuous- time state equation and we will track asymptotically any step reference input:

$$\dot{X} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \qquad \text{and} \qquad y = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} X \qquad ---\rightarrow (1)$$

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In this problem, we have to find the feed forward gain (ρ) and the state feedback gain (k) so that the resulting system has eigenvalues (-2) and ($-1 \pm j1$) and will track asymptotically any step reference input.

3. Compute the state feedback gain By Calculation:

$$\Delta(\lambda) = \begin{bmatrix} \lambda-1 & -1 & 2 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 \quad ---\rightarrow (2)$$

So, $\alpha_1 = -3, \alpha_2 = 3, \alpha_3 = -1$

$$\Delta_f(\lambda) = (\lambda - P1)(\lambda - P2)(\lambda - P3) \quad ---\rightarrow (3)$$

$$\Delta_f(\lambda) = (\lambda+2)(\lambda+1+j)(\lambda+1-j)$$
$$= \lambda^3 + 4\lambda^2 + 6\lambda + 4$$

So, $\bar{\alpha}_1 = 4, \bar{\alpha}_2 = 6, \bar{\alpha}_3 = 4 \quad ---\rightarrow (4)$

$$\bar{k} = [4-(-1) \quad 6-3 \quad 4-(-3)]$$
$$= [5 \quad 3 \quad 7]$$

$$AB = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad A^2B = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad ---\rightarrow (5)$$

The conrollability matrix: $C = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$\bar{C}^{-1} = \begin{bmatrix} \alpha_2 & \alpha_1 & 1 \\ \alpha_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad P^{-1} = C\bar{C}^{-1} = \begin{bmatrix} 4 & -4 & 1 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad ---\rightarrow (6)$$

$P = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix}$, now we can find the feedback gain as:

$$k = \bar{k}P = [5 \quad 3 \quad 7] \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix} = [15 \quad 47 \quad -8] \quad ---\rightarrow (7)$$

In order to find the feed forward gain, we have to compute the transfer function from (r) to (y):

$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \rho \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \beta_3}{s^n + \bar{\alpha}_1 s^{n-1} + \bar{\alpha}_2 s^{n-2} + \bar{\alpha}_3} \quad ----\rightarrow (8)$$

$$\bar{C} = CP^{-1} = [2 \quad 0 \quad 0]\begin{bmatrix} 4 & -4 & 1 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = [8 \quad -8 \quad 2] = [\beta_3 \ \ \beta_2 \ \ \beta_1]$$

$$\therefore \ \hat{g}_f(s) = p.\frac{2s^2 - 8s + 8}{s^3 + 4s^2 + 6s + 4}$$

$$\hat{g}_f(0) = 1 = p.\frac{8}{4}$$

$$\therefore The\ feed\ forward\ gain \qquad p = \frac{4}{8} = 0.5$$

we can find the transfer function by using Simulation to simulate our system as:

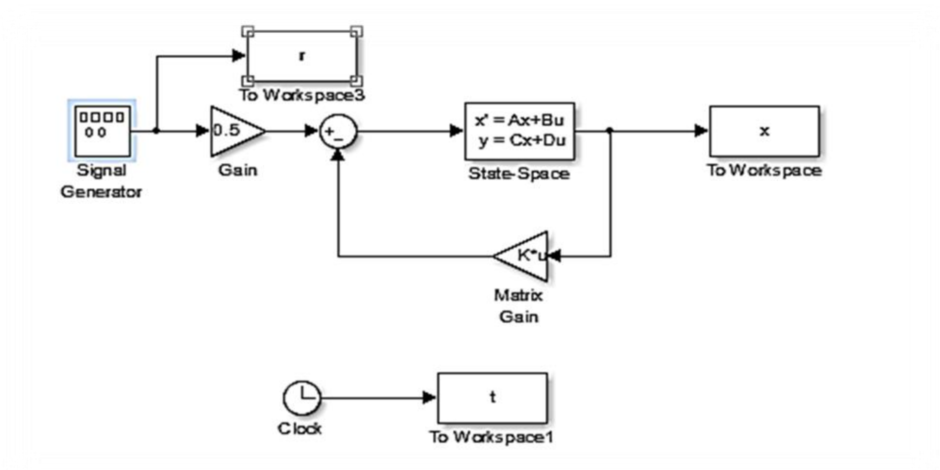

Figure 1. using Simulation to simulate our system

In order to plot the relationship between the reference and the output in the same figure to show the tracking between them, we used the next command:

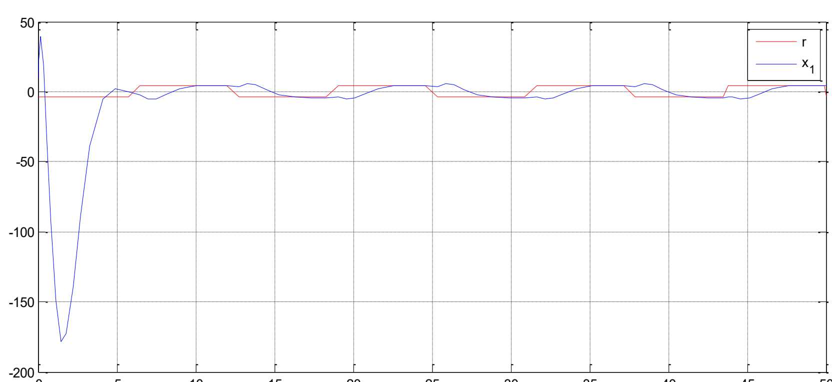

Figure 2. the reference (r) and output (x_1) with 0.08 frequency

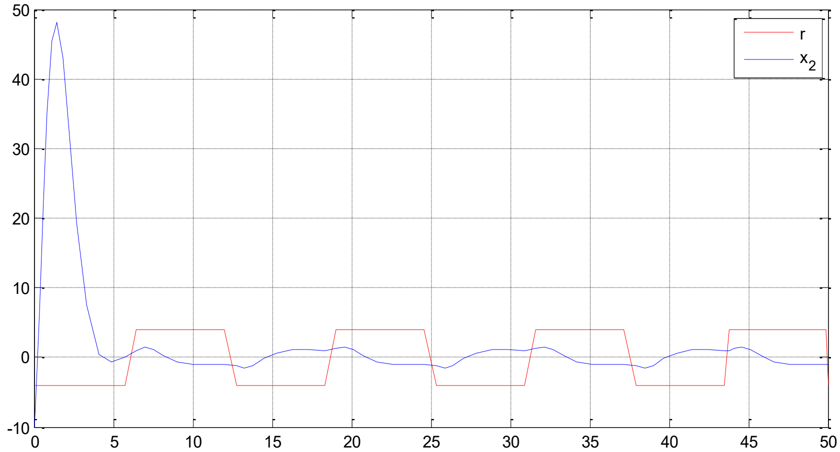

Figure 3. the reference (r)and output (x_2) with 0.08 frequency

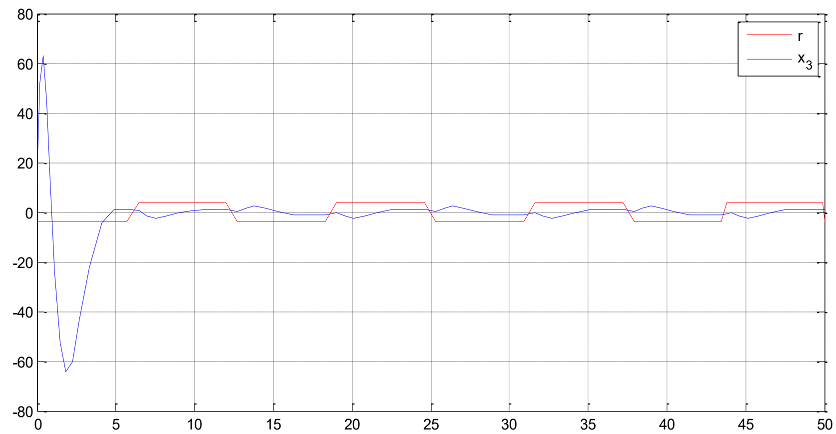

Figure 4. the reference (r) and output (x_3) with 0.08 frequency

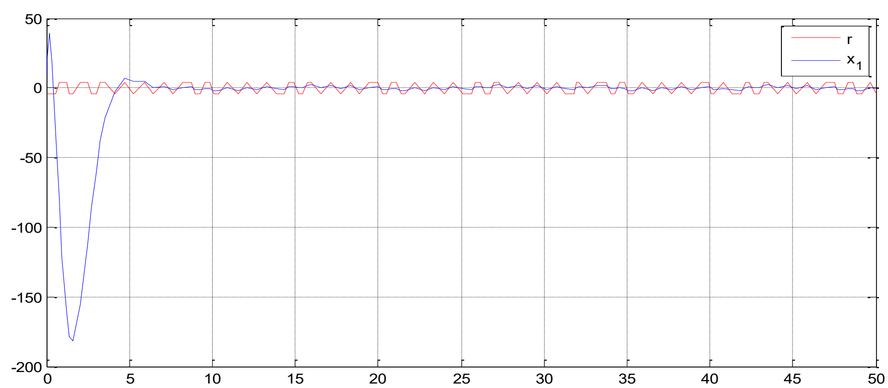

Figure 5. the reference (r) and output (x_1) with 0.8 frequency

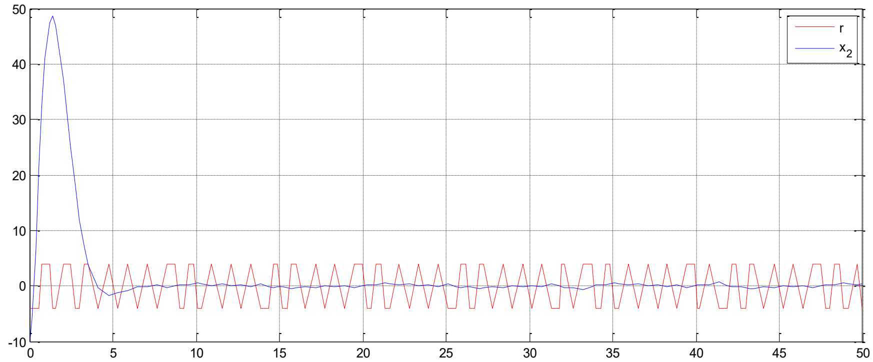

Figure 6. the reference (r) and output (x_2) with 0.8 frequency

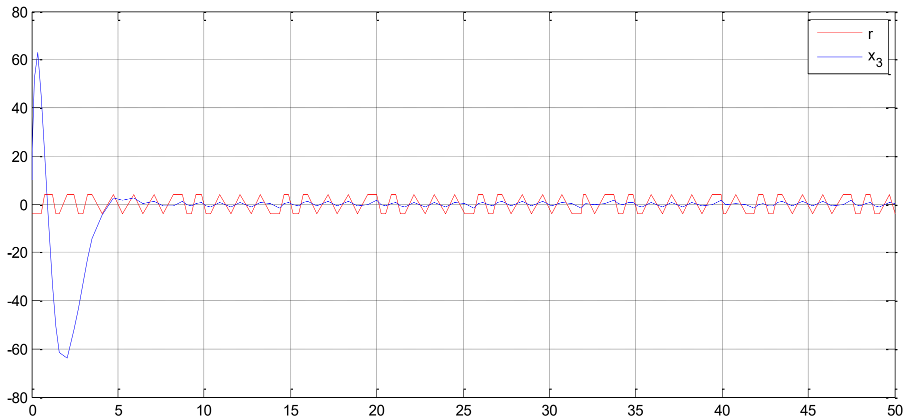

Figure 7. the reference (r) and output (x_3) with 0.8 frequency

From the figures above, it is very clear that the period is decreased by increasing the frequency. Also, the system has become more fluctuating (Weibel and Kaper, 1995) because of the decreasing of period (Levi and Weckesser, 1995). The system's output clearly tracking the input, but also does not reach the amplitude of the input (Fontich and Simo, 1990).

4. Pendulum on a cart

For choose reasonable values for the system's parameters and using MATLAB to simulate the system using the original nonlinear equations "Open loop system" (Melnikov et al, 1963) and then using the linearized model of the system to design a state-feedback controller that balances the pendulum and regulates the cart's position to zero (Leonard and Krishnaprasad, 1995).

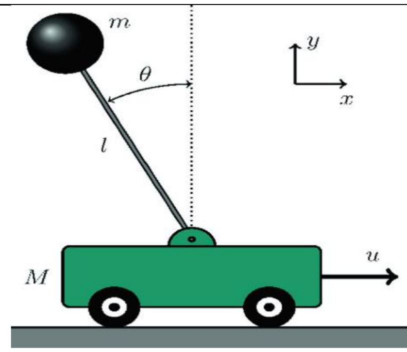

Figure 8. Pendulum on a cart

Frist, we have to work on the first case (1) to come up with the state form:

$$(M+m)\ddot{y} - m\,l\,sin\theta\,\dot{\theta}^2 + m\,l\,cos\theta\,\ddot{\theta} = F \qquad ---\rightarrow (9)$$
$$m\,\ddot{y}\,cos\theta + m\,l\,\ddot{\theta} = m\,g\,sin\theta \qquad ---\rightarrow (10)$$

The second case (2)

$$m\,l\,\ddot{\theta} = m\,g\,sin\theta - m\,\ddot{y}\,cos\theta$$

Putting this into equation (1)

$$(M+m)\ddot{y} - m\,l\,sin\theta\,\dot{\theta}^2 + m\,g\,cos\theta\,sin\theta - m\,\ddot{y}\,cos^2\theta = F$$
$$(M+m-m\,cos^2\theta)\ddot{y} = F + m\,l\,sin\theta\,\dot{\theta}^2 + m\,g\,cos\theta\,sin\theta \qquad ---\rightarrow (11)$$

The second case (2)

$$\ddot{y} = \frac{g\,sin\theta - l\,\ddot{\theta}}{cos\theta}$$

Putting this into the first case (1)

$$(m\,l\,cos^2\theta - (M+m)l)\ddot{\theta} = F\,cos\theta - (M+m)\,g\,sin\theta + m\,l\,cos\theta\,sin\theta\,\dot{\theta}^2 \qquad ---\rightarrow (12)$$

From The third case (3) and the fourth case (4) we have the following:

$$\ddot{y} = \frac{F + m\,l\,sin\theta\,\dot{\theta}^2 + m\,g\,cos\theta\,sin\theta}{(M+m-m\,cos^2\theta)} \qquad ---\rightarrow (13)$$

$$\ddot{\theta} = \frac{F\,cos\theta - (M+m)\,g\,sin\theta + m\,l\,cos\theta\,sin\theta\,\dot{\theta}^2}{m\,l\,cos^2\theta - (M+m)l} \qquad ---\rightarrow (14)$$

We can come up with the state form above as:

$Let \quad x_1 = \theta \qquad x_2 = \dot{\theta} = \dot{x}_1 \qquad x_3 = y \qquad x_4 = \dot{y} = \dot{x}_3$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{y}\\ \ddot{y}\end{bmatrix}=\begin{bmatrix} x_2\\ \dfrac{F\,cosx_1-(M+m)\,g\,sinx_1\;+m\,l\,cosx_1\,sinx_1\,x_2{}^2}{m\,l\,cos^2x_1-(M+m)l}\\ x_4\\ \dfrac{F+m\,l\,sinx_1\,x_2{}^2+m\,g\,cosx_1\,sinx_1}{(M+m-m\;cos^2x_1)}\end{bmatrix}$$

When the actuating force is set to F=0, then:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{y}\\ \ddot{y}\end{bmatrix}=\begin{bmatrix} x_2\\ \dfrac{-(M+m)\,g\,sinx_1\;+m\,l\,cosx_1\,sinx_1\,x_2{}^2}{m\,\mathrm{l}\,\mathrm{cos}^2\mathrm{x}_1-(\mathrm{M}+\mathrm{m})\mathrm{l}}\\ \mathrm{x}_4\\ \dfrac{\mathrm{m\,l\,sinx_1\,x_2{}^2+m\,g\,cosx_1\,sinx_1}}{\mathrm{(M+m-m\;cos^2x_1)}}\end{bmatrix}$$

5. OPEN-LOOP- SYSTEM SIMULATION:

Then we plotted the four states versus the time as following:

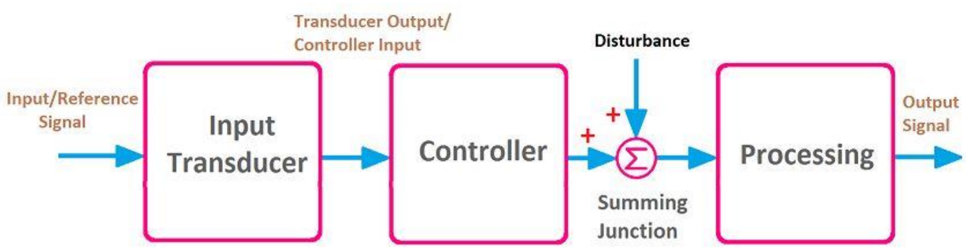

Figure 9. shown Open-Loop- system

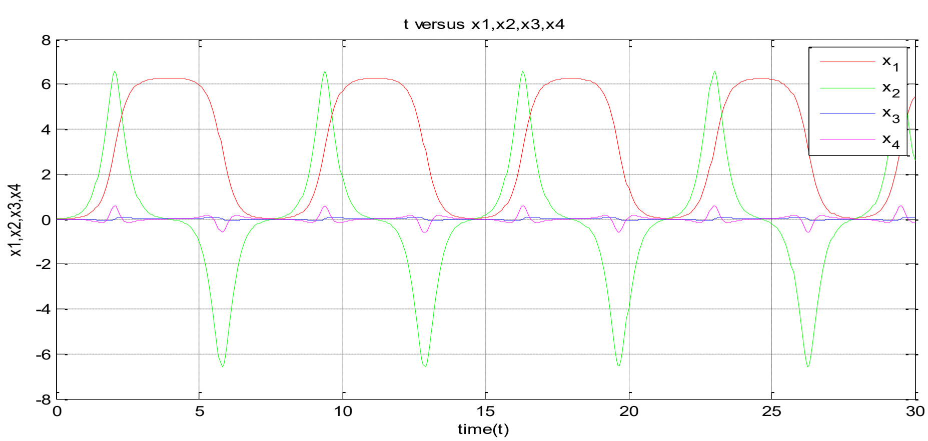

Figure 10. shown the four states versus the time

In this part, we have to design state Feedback controller that balances the pendulum and regulates the cart's position to zero and we can use the linearized model that we found.

$$\ddot{y} = \frac{1}{M}F - \frac{mg}{M}x_1 \quad ---\rightarrow (14)$$

$$\ddot{\theta} = \frac{(M+m)g}{l\,M}x_1 - \frac{1}{lM}F \quad ---\rightarrow (15)$$

Let $x_1 = \theta \qquad x_2 = \dot{\theta} = \dot{x}_1 \qquad x_3 = y \qquad x_4 = \dot{y} = \dot{x}_3$

So, the state form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{y}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{(M+m)g}{l\,M} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{mg}{M} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{lM}\\ 0\\ \frac{1}{M}\end{bmatrix}F$$

$$y = [0 \quad 0 \quad 1 \quad 0]x$$

Also, in this part we plotted the four states versus the time as following:

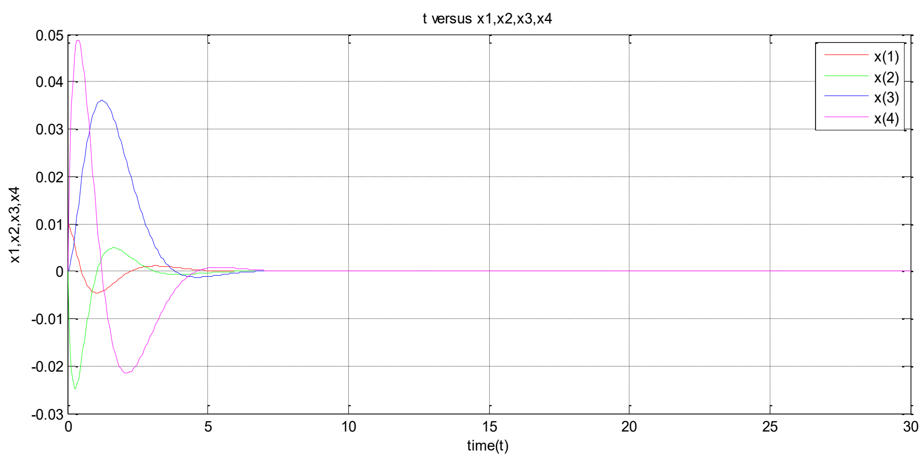

Figure 11. shown the four states versus the time

Results:

From continuous- time state equation and we will track asymptotically any step reference input the system has become more fluctuating because of the decreasing of period and the system's output clearly tracking the input, but also does not reach the amplitude of the input .And, By using (MATLAB) to simulate the system using the original nonlinear and then using the linearized model of the system to design a state-feedback controller that balances the pendulum and regulates the cart's position to zero.

Conclusion

According to the results we got from this paper the focus here was on state feedback and the simulation of the system once the controller parameters had been determined, and the inverted pendulum is relatively easy to visualize. Moreover, consider the inverted pendulum on a cart and choose reasonable values for the system's parameters and using (MATLAB) to simulate the system using the original nonlinear equations

(Open loop system) and then using the linearized model of the system to design a state-feedback controller that balances the pendulum and regulates the cart's position to zero.

Conflict of Interest Statement

The authors declare no potential conflicts of interest

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