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Generalized of Some Basic Operations on Fuzzy sets

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Abstract:

This study deals with the basic concept of fuzzy sets. This study reviews the operations on fuzzy sets and numbers, it also discusses how to relate fuzzy sets and fuzzy numbers. It also focuses on the main idea of fuzzy relations and the projection of fuzzy relations.

Keywords: *Fuzzy sets, operations of Fuzzy sets, Fuzzy numbers, Relations, Projections.*

1 INTRODUCTION:

A class of objects of degrees of membership with a continuum is defined as fuzzy sets [1] and [2]. For instance, a set is described by a membership function that appoints for every object a membership degree between 0 and 1 [3]. The notions of all operations such as union, intersection, complement, relation, convexity, etc. are extended to such sets, and various properties of these notions are established in the context of fuzzy sets. Fuzziness occurs when the boundary of information is not unique. Fuzzy sets as an extension of the classical notation for a more detailed look at [4], [5], [6], and [7].

The classical set theory allows the membership of classically the set-in binary terms. Fuzzy set theory is giving permission membership function valued in the interval [0,1]. In particular, a separation theorem for convex fuzzy sets is proved without the need for the fuzzy sets to be disjoint [1] and [4]. In most cases, the classes of objects that occur in the real world do not have well-defined criteria for membership. For example, the class of animal dogs, horses, birds, etc. as their members, and clearly such objects as fluids, plants, rocks, etc., are not disjoint. However, objects such as viruses, bacteria, and starfish which have an ambiguous status in relation to the class of animals.

In spite of this fact remains that such imprecisely defined "classes" play an important role in human thought, especially in the areas of pattern recognition, information transmission, and abstraction. The aim of this study is to figure out some of the basic properties and implications of a concept of fuzzy sets that might be useful for dealing with classes.

The paper is organized as follows: The concept of fuzzy set and some related definitions are discussed in the next section. $\boldsymbol{\alpha}$-Cuts and Decomposition Principle, Cartesian Product, fuzzy numbers with their operations are given in Section 3. In Section 4, Fuzzy Numbers in the set of integers is discussed. Section 5, Fuzzy relations with some applications are defined. Section 6, some conclusions are drawn and suggestions for future research are given.

2 Fuzzy Sets

To clarify the fuzzy sets, let us introduce some examples such as Words like young, tall, good, or high are fuzzy. There is no single quantitative value that defines the term young. For some people, age 25 is young, and for others, age 35 is young. The concept of young has no clear boundary. Age

35 has some possibility of being young and usually depends on the context in which it is being considered. Fuzzy set theory is an extension of classical set theory where elements have a degree of membership. The meaning of fuzzy is not clear, let us discusses fuzzy set $x =$ All students in class, $S =$ All $good$ students, $S = \{x\colon \mathrm{M}_S(x)\colon x \in X\}$. Then, the fuzzy set is

$$\underset{\sim}{S} = \{\frac{\mathrm{M}_S(x_1)}{x_1}, \frac{\mathrm{M}_S(x_2)}{x_2}, \dots\}$$

Hence $\mathrm{M}_S(x)$ is measurement of good students.

2.1 Fuzzy sets membership functions

A fuzzy (sub)set A on the universe X is a set defined by a membership function μ_A representing a mapping. $\mu_A\colon X \longrightarrow [0,1]$, Here, the value of $\mu_A(x)$ for the fuzzy set A is called the membership value of $x \in X$. The membership value is indicated to the degree of x belonging to the fuzzy set A.

2.2 Expression of fuzzy:

• Discrete expression (when the universe is finite): Let the universe X be $X = \{x_1, x_2, \dots., x_n\}$. Then, a fuzzy set A on X can be represented as follows:

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i$$

(Membership value of the first element)/(the value of the first element)+ (Membership value of the second element)/(the value of the second element)+…+ (Membership value of the nth element)/(the value of the nth element).

Continuous expression: (when the universe is infinite): When the universe X is an infinite set, a fuzzy set A on X can be represented as follows.

$$A = \int \mu_A(x_i)/x_i = \int (\text{Membership function/Element variable})$$

$\int$ is used as an extension of $\sum to$ the continuous world, and it has no connection with the integral. There are two other rules for the discrete expression:

i- When the grade of membership for an element x' is zero, that is $\mu_A(x') = 0$, we do not write $0/x'$ but we can omit the term.

ii- If there are several values assigned to one element of the universe, we can take the maximum value to represent the membership value.

Example 2.1: Trapezoidal fuzzy sets:

The trapezoidal fuzzy set can be expressed by the infinite expression:

$$B = \int_{-4}^{-2} \left(\frac{4+x}{2}\right)/x + \int_{-2}^{2} 1/x + \int_{2}^{4} \left(\frac{4-x}{2}\right)/x$$

Next, let us think finite expression. If X is given as

$$X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

The finite expression of the fuzzy set B will be

$$B = 0.5/{-3} + 1/{-2} + 1/{-1} + 1/0 + 1/1 + 1/2 + 0.5/3.$$

2.3 Normal, convex, and scalar cardinality

Let A be a fuzzy on the universe X. A normal fuzzy set, a convex fuzzy set, and the cardinality of a fuzzy set are defined as follows.

- Normal fuzzy set: the fuzzy set A is normal if $\max_{x\in X}\mu_A(x) = 1$.
- Convex fuzzy set: the fuzzy set A is convex if $\forall x_1 \in X, \forall x_2 \in X, \forall \lambda \in [0,1]$

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

While, if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \leq \mathrm{Max}(\mu_A(x_1), \mu_A(x_2))$
It is called concave fuzzy set in this case.

- Scalar Cardinality: when X is a finite set, the cardinality of fuzzy set A on X is defined by

$$|A| = \sum_{x\in X} \mu_A(x)$$

- Relative cardinality: the relative cardinality of fuzzy set A on X is defined by

$$||A|| = \frac{|A|}{|X|}$$

Where $|A|$ is the cardinality of A and $|X|$ is the cardinality of the universe X.

Example 2.2: Suppose two fuzzy sets A and B are given by the following arrays.

Name of array A

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$
0.5	0.9	0.7	1	0.7	0.3

Name of array B

$B[0]$	$B[1]$	$B[2]$	$B[3]$	$B[4]$	$B[5]$
0	0.4	0.8	0.7	0.2	0

To judge if a fuzzy set is normal, we find the maximum value of the array and check if it is equal to1. Because the maximum value of the array A is 1, fuzzy set A is normal. On the other hand, because the maximum value of the array B is 0.8, fuzzy set B is not normal.

The judgment of convexity is not as straightforward as that normality. To check the convexity of A, we start from the first (leftmost) element of the array, we compare the sequentially. We observe the relationship for the first few elements as

$$A[i] < A[i+1], i = 0,1,..$$

If we preceding inequality no longer holds at an i-th element we store the subscript i, we note that $A[1] > A[2]$. Therefore, we store the subscript 1.

Next, from the last (rightmost) element of the array we compare the adjacent element in the reverse direction. We observe for the first few elements of the search the following inequality holds:

$$A[i] \geq A[i+1], i = k, k-1, k-2, \ldots$$

Here, k is the largest subscript.

we note that $A[2] < A[3]$. Therefore, we store the subscript 3. If the subscripts stored in the preceding operations are same, the fuzzy set is convex. Here, there are two subscripts are different in array A, fuzzy set A is not convex, whereas the fuzzy set B is convex.

Cardinality of fuzzy sets A and B are:

$$|A| = \sum_{x\in X} \mu_A(x) = 0.5 + 0.9 + 0.7 + 1 + 0.7 + 0.3 = 3.2 \ \& \ |B| = \sum_{x\in X} \mu_B(x) = 2.1$$

The relative cardinality of the fuzzy sets And B are

$$||A|| = \frac{3.2}{6} = 0.533 \ \& \ ||B|| = \frac{2.1}{6} = 0.35$$

2.4 Fundamental Operation of Fuzzy Sets-Union, Intersection and Complement

- Union of fuzzy sets A and B: Union $A \cup B$ of fuzzy sets A and B is a fuzzy set defined by the membership function: $\mu_{A\cup B} = \mu_A \vee \mu_B$

  Where $\mu_A \vee \mu_B = \begin{cases} \mu_A & \mu_A \geq \mu_B \\ \mu_B & \mu_A < \mu_B \end{cases}$, $\mu_A \vee \mu_B$ can be written as $\max\{\mu_A, \mu_B\}$.
- Intersection of fuzzy sets A and B: Intersection $A \cap B$ of fuzzy sets A and B is a fuzzy set defined by the membership function: $\mu_{A\cap B} = \mu_A \wedge \mu_B$

  Where $\mu_A \wedge \mu_B = \begin{cases} \mu_A & \mu_A \leq \mu_B \\ \mu_B & \mu_A > \mu_B \end{cases}$, $\mu_A \wedge \mu_B$ can be written as $\min\{\mu_A, \mu_B\}$
- Complement of fuzzy set A: complement $\bar{A}$ of fuzzy set A is a fuzzy set defined by the membership function: $\mu_{\bar{A}} = 1 - \mu_A$

Example 2.3:

Suppose two fuzzy sets A and B are given by the following arrays. Name of array A

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	$A[7]$	$A[8]$	$A[9]$
0	0.1	0.6	1	0.6	0.1	0	0	0	0

Name of array B

$B[0]$	$B[1]$	$B[2]$	$B[3]$	$B[4]$	$B[5]$	$B[6]$	$B[7]$	$B[8]$	$B[9]$
0	0	0.1	0.4	0.7	1	0.7	0.4	0.1	0

Then,

$A \cup B =$	0	0.1	0.6	1	0.7	1	0.7	0.4	0.1	0
$A \cap B =$	0	0	0.1	0.4	0.6	0.1	0	0	0	0

$\mu_{\bar{A}} =$	1	0.9	0.4	0	0.4	0.9	1	1	1	1
$\mu_{\bar{B}} =$	1	1	0.9	0.6	0.4	0	0.3	0.6	0.9	1

2.5 Properties of fuzzy sets

Let A, B and C be fuzzy sets on the universe X.

i-properties valid for both fuzzy and crisp sets:

- Idempotent law: $A \cup A = A$ & $A \cap A = A$
- Communicative law: $A \cup B = B \cup A$ & $A \cap B = B \cap A$
- Associative law: $A \cup (B \cup C) = (A \cup B) \cup C$ & $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- The law of double negation: $A = \bar{\bar{A}}$
- De Morgan's law: $\overline{A \cup B} = \bar{A} \cap \bar{B}$ & $\overline{A \cap B} = \bar{A} \cup \bar{B}$

ii- Properties valid for crisp sets. but in general, not valid for fuzzy sets.

- The law of exclude middle: $A \cup \bar{A} \neq X$
- The law of contradiction: $A \cap \bar{A} \neq \emptyset$, Where $\emptyset$ means an empty set.

2.6 Equality and inclusion of fuzzy sets:

Let A and B be fuzzy sets on the universe X.

- Equality of fuzzy sets: the equality of fuzzy sets A and B is defined as

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$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \qquad \forall x \in X$$

- Inclusion of fuzzy sets: the inclusion of fuzzy sets A in B, or A being a subset of B, is defined as: $A \subset B \Leftrightarrow \mu_A \leq \mu_B,\ \ \forall x \in X$

Fuzzy sets A and B are equal when their membership values are identical. Similarly, A is included in B when all the membership values of fuzzy set B are equal to or larger than the corresponding membership values of A.

Example 2.4: Suppose two fuzzy sets A and B are given by the following arrays.

Name of array A

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	$A[7]$	$A[8]$	$A[9]$
0	0.1	0.6	1	0.6	0.1	0	0	0	0

Name of array B

$B[0]$	$B[1]$	$B[2]$	$B[3]$	$B[4]$	$B[5]$	$B[6]$	$B[7]$	$B[8]$	$B[9]$
0	0.1	0.6	1	0.6	0.1	0	0	0	0

To judge the equality of the preceding fuzzy sets we compare each corresponding element and check if they are equal: We notice: $A[i] = B[i], i = 0,1,2,\dots,9$

Therefore, $A = B$. Also, if two fuzzy sets A and B are given by the following arrays. Name of array A

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	$A[7]$	$A[8]$	$A[9]$
0	0.1	0.5	0.9	0.5	0	0	0	0	0

Name of array B

$B[0]$	$B[1]$	$B[2]$	$B[3]$	$B[4]$	$B[5]$	$B[6]$	$B[7]$	$B[8]$	$B[9]$
0	0.1	0.6	1	0.6	0.1	0	0	0	0

We notice that: $A[i] \leq B[i], i = 0,1,2,\dots,9$. Therefore, $A \subset B$.

3 α-Cuts and Decomposition Principle

- Consider a fuzzy set A we can define the following **α-Cuts. Strong α-cut**: $A_\alpha = \{x|\mu_A(x) > \alpha\}, \alpha \in [0,1)$. **Weak α-cut:** $A_{\overline{\alpha}} = \{x|\mu_A(x) \geq \alpha\}, \alpha \in (0,1]$. Weak α-cuts are sometimes **called α-level sets**.
- Decomposition principle: using α-cuts we can here decompose a membership function $\mu_A(x)$ into an infinite number of rectangular membership functions $\left(\alpha \wedge X_{A_\alpha}(x) \text{or } \alpha \wedge X_{A_{\overline{\alpha}}}(x)\right)$. When we aggregate these rectangular membership functions and apply max -operation- the original fuzzy set A can be obtained:

$$\mu_A(x) = \max_{\alpha \in [0,1)} [\alpha \wedge X_{A_\alpha}(x)] = \max_{\alpha \in (0,1]} [\alpha \wedge X_{A_{\overline{\alpha}}}(x)]$$

Where, X_{A_α} is a characteristic equation of the set A_α.

Example 3.1: Suppose a fuzzy set A defined by the discrete expression:

$$A = 0.2/1 + 0.5/2 + 0.7/3 + 1/4 + 0.8/5 + 0.4/6 + 0.2/7.$$

If we apply Weak α-cut for $\}, \alpha \in (0,1]$, from 0.1 to 1with the step width 0.1, we get the following α-cut:

$$A_{\overline{0.1}} = A_{\overline{0.2}} = \{1,2,3,4,5,6,7\}, A_{\overline{0.3}} = A_{\overline{0.4}} = \{2,3,4,5,6\}, A_{\overline{0.5}} = \{2,3,4,5\},$$
$$A_{\overline{0.6}} = A_{\overline{0.7}} = \{3,4,5\}, A_{\overline{0.8}} = \{4,5\}, A_{\overline{0.9}} = A_{\overline{1}} = \{4\}.$$

Now let us try to reconstruct the fuzzy set A using these α-cuts. First, we rewrite the α-cuts by discrete expression of fuzzy sets as follows.

$A_{\overline{0.1}} = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7.$

$A_{\overline{0.2}} = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7.$

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$A_{\overline{0.3}} = 1/2 + 1/3 + 1/4 + 1/5 + 1/6.$
$A_{\overline{0.4}} = 1/2 + 1/3 + 1/4 + 1/5 + 1/6.$
$A_{\overline{0.5}} = 1/2 + 1/3 + 1/4 + 1/5.$
$A_{\overline{0.6}} = 1/3 + 1/4 + 1/5.$
$A_{\overline{0.7}} = 1/3 + 1/4 + 1/5.$
$A_{\overline{0.8}} = 1/4 + 1/5.$
$A_{\overline{0.9}} = 1/4.$
$A_{\overline{1}} = 1/4.$

Here, because $A_{\overline{0.1}}$ is given as $A_{\overline{0.1}} = \{1,2,3,4,5,6,7\}$,bif we focus on the value of the characteristic equation, we note that $X_{A_{\overline{\alpha}}}(1) = X_{A_{\overline{\alpha}}}(2) = \cdots = X_{A_{\overline{\alpha}}}(7) = 1$

Next, we calculate $\alpha \wedge X_{A_{\overline{\alpha}}}(x)$. Assume $x_1 = 1, x_2 = 2, \ldots, x_7 = 7$ or $x_i = i,\ i = 1, \ldots, 7$

If we denote a fuzzy set $A_{\frac{*}{\alpha}}$ that has $\alpha \wedge X_{A_{\overline{\alpha}}}(x)$as the membership value, we can calculate $A_{\frac{*}{\alpha}}$ as follows:

$$A_{\frac{*}{0.1}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.1}}}(x_i)\right]/x_i\right) = \left[0.1 \wedge X_{A_{\overline{0.1}}}(x_1)\right]/x_1 + \cdots + \left[0.1 \wedge X_{A_{\overline{0.1}}}(x_7)\right]/x_7$$
$$= 0.1/1 + 0.1/2 + 0.1/3 + 0.1/4 + 0.1/5 + 0.1/6 + 0.1/7$$

$$A_{\frac{*}{0.2}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.2}}}(x_i)\right]/x_i\right) = \left[0.1 \wedge X_{A_{\overline{0.2}}}(x_1)\right]/x_1 + \cdots + \left[0.1 \wedge X_{A_{\overline{0.2}}}(x_7)\right]/x_7$$
$$= 0.2/1 + 0.2/2 + 0.2/3 + 0.2/4 + 0.2/5 + 0.2/6 + 0.2/7$$

$$A_{\frac{*}{0.3}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.3}}}(x_i)\right]/x_i\right) = \left[0.1 \wedge X_{A_{\overline{0.3}}}(x_1)\right]/x_1 + \cdots + \left[0.1 \wedge X_{A_{\overline{0.3}}}(x_7)\right]/x_7$$
$$= 0.3/2 + 0.3/3 + 0.3/4 + 0.3/5 + 0.3/6$$

$$A_{\frac{*}{0.4}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.4}}}(x_i)\right]/x_i\right) = \left[0.1 \wedge X_{A_{\overline{0.4}}}(x_1)\right]/x_1 + \cdots + \left[0.1 \wedge X_{A_{\overline{0.4}}}(x_7)\right]/x_7$$
$$= 0.4/2 + 0.4/3 + 0.4/4 + 0.4/5 + 0.4/6$$

$$A_{\frac{*}{0.5}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.5}}}(x_i)\right]/x_i\right) = \left[0.1 \wedge X_{A_{\overline{0.5}}}(x_1)\right]/x_1 + \cdots + \left[0.1 \wedge X_{A_{\overline{0.5}}}(x_7)\right]/x_7$$
$$= 0.5/2 + 0.5/3 + 0.5/4 + 0.5/5$$

$$A_{\frac{*}{0.6}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.6}}}(x_i)\right]/x_i\right) = 0.6/3 + 0.6/4 + 0.6/5$$

$$A_{\frac{*}{0.7}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.7}}}(x_i)\right]/x_i\right) = 0.7/3 + 0.7/4 + 0.7/5$$

$$A_{\frac{*}{0.8}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.8}}}(x_i)\right]/x_i\right) = 0.8/4 + 0.8/5$$

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$$A_{\frac{*}{0.9}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{0.9}}}(x_i)\right]/x_i\right) = 0.9/4 \ \& \ A_{\frac{*}{1.0}} = \sum_{i=1}^{7}\left(\left[0.1 \wedge X_{A_{\overline{1.0}}}(x_i)\right]/x_i\right) = 1.0/4$$

Calculating the union of the preceding the fuzzy sets, we get the original fuzzy set A

$$\bigcup_{\alpha\in(0,1]} A_{\frac{*}{\alpha}} = \bigcup_{\alpha\in(0,1]} \sum_{i=1}^{7}\left(\left[\alpha \wedge X_{A_{\overline{\alpha}}}(x_i)\right]/x_i\right)$$
$$= 0.2/1 + 0.5/2 + 0.7/3 + 1/4 + 0.8/5 + 0.4/6 + 0.2/7 = A$$

This can also be written as $\mu_A(x) = \max_{\alpha\in(0,1]}[\alpha \wedge X_{A_{\overline{\alpha}}}(x)]$

Example 3.2:

Suppose a fuzzy set is given as

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	$A[7]$	$A[8]$
0	0.1	0.6	1	0.5	0.1	0	0	0

The α-cuts are calculated by processing elements of A so that, If $\alpha < A[i]$, $A[i] = 1, i = 0,1, \dots ,8$, otherwise $A[i] = 0$.

For $\alpha = 0.2$, $\alpha = 0.6$, and $\alpha = 0.8$, resultant

$A_{\overline{0.2}}$ will be

0	0	1	1	1	0	0	0	0

$A_{\overline{0.6}}$ will be

0	0	1	1	0	0	0	0	0

$A_{\overline{0.8}}$ will be

0	0	0	1	0	0	0	0	0

3.1 Extension Principle

Consider a mapping from a set X to another Y such as $f: X \longrightarrow Y$. Here, let A be a subset of X. Then, $f(A) = \{y|y = f(x), x \in A\}$. Is called the image of A by f. Note that $f(A)$is a subset of Y. Similarly, let B be subset of Y. Then, $f^{-1}(B) = \{x|f(x) = y, y \in B\}$. Is called an inverse image of B by f. $f^{-1}(B)$.

• **Extend mapping** $f: X \longrightarrow Y$ to relate fuzzy set A on X to fuzzy set B on Y:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{y=f(x)} \mu_A(x) & f^{-1}(y) \neq \emptyset \\ 0 & f^{-1}(y) = \emptyset \end{cases}$$

When $f$is one to one mapping, we can write the preceding relation simply: $\mu_{f(A)}(y) = \mu_A(x)$

Example 3.3: If $y = 3x + 2$. Using fuzzy sets and extension principle, let A be the fuzzy set that gives about 4 such that: $A = 0.5/3 + 1/4 + 0.5/5$. Also, define $x_1 = 3$ $x_2 = 4$, $x_3 = 5$ so that: $y_i = 3x_i + 2, i = 1,2,3$. To get $f(A)$ for fuzzy set A as follow

$$f(A) = \sum_{i=1}^{3} \mu_{f(A)}(y_i)\,/y_i = \sum_{i=1}^{3} \mu_{f(A)}(3x_i + 2)\,/(3x_i + 2)$$
$$= 0.5/11 + 1/14 + 0.5/17 = about\ 14$$

3.2 Cartesian Product

• Cartesian product: let $x_1, \dots, x_n$be the elements of $X_1, \dots, X_n$. The set of all combination of $(x_1, \dots, x_n)$ is called the Cartesian product of $X_1, \dots, X_n$ and is denoted by $X_1 \times \dots \times X_n$.

• Cartesian product of fuzzy sets: let $X_1 \times \dots \times X_n$ of the universe $X_1, \dots, X_n$ and $A_1, \dots, A_n$ fuzzy sets on $X_1, \dots, X_n$. The Cartesian of the fuzzy sets $A_1, \dots, A_n$ can be defined by

$$A_1 \times \dots \times A_n = \int_{X_1 \times \dots \times X_n} \min(\mu_A(x_1), \dots, \mu_A(x_n))/(x_1, \dots, x_n)$$

On the universe $X_1 \times \dots \times X_n$.

Example 3.4:

If the fuzzy sets A and B are given as:

$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$ and $B(x) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$\min\{\mu_A(x_1), \mu_B(y_1)\} = 0.2, \min\{\mu_A(x_1), \mu_B(y_2)\} = 0.2, \min\{\mu_A(x_1), \mu_B(y_3)\} = 0.2$$
$$\min\{\mu_A(x_2), \mu_B(y_1)\} = 0.3, \min\{\mu_A(x_2), \mu_B(y_2)\} = 0.3, \min\{\mu_A(x_2), \mu_B(y_3)\} = 0.3$$
$$\min\{\mu_A(x_3), \mu_B(y_1)\} = 0.5, \min\{\mu_A(x_3), \mu_B(y_2)\} = 0.5, \min\{\mu_A(x_3), \mu_B(y_3)\} = 0.3$$
$$\min\{\mu_A(x_4), \mu_B(y_1)\} = 0.6, \min\{\mu_A(x_4), \mu_B(y_2)\} = 0.6, \min\{\mu_A(x_4), \mu_B(y_3)\} = 0.3$$

$$A \times B = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix}$$

3.3 Fuzzy numbers

• fuzzy numbers: if a fuzzy set A on the universe R of real numbers satisfies the following conditions, we call it a fuzzy number

i. A is a convex fuzzy set;
ii. There is only one x_0 that satisfies $\mu_A(x_0) = 1$
iii. μ_A is continuous in an interval.

• Flat fuzzy numbers: if a fuzzy number A satisfies the following condition,

$$m_1, m_2 \in R, \qquad m_1 < m_2, \qquad \mu_A(x) = 1 \quad \forall x \in [m_1, m_2]$$

3.3.1 Operation of Fuzzy numbers based on the extension principle

The operation $*$ of real numbers can be extended to fuzzy numbers A and B on the universe X such as

$$\mu_{A \circledast B}(z) = \sup_{x*y}[\mu_A(x) \wedge \mu_B(x)]$$

If we rewrite the preceding expression using fuzzy sets, we get

$$A \circledast B = \int_{X \times X} [\mu_A(x) \wedge \mu_B(x)]/(x * y)$$

Where $x, y, z \in X$

3.3.2 Arithmetic of fuzzy numbers

• **Addition:** $\mu_{A \oplus B}(z) = \sup_{x+y}[\mu_A(x) \wedge \mu_B(x)]$

For any two intervals $A + B = [a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

• **Subtraction:** $\mu_{A \ominus B}(z) = \sup_{x-y}[\mu_A(x) \wedge \mu_B(x)]$

For any two intervals

$$A - B = [a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$$

• **Multiplication:** $\mu_{A\circledast B}(z) = \sup_{x\times y}[\mu_A(x) \wedge \mu_B(x)]$

For any two intervals

$$A.B = [a_1, a_2][b_1, b_2] = [\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)]$$

• **Division:** $\mu_{A/B}(z) = \sup_{x/y}[\mu_A(x) \wedge \mu_B(x)]$

For any two intervals $A: B = \frac{A}{B} = \frac{[a_1,a_2]}{[b_1,b_2]} = [a_1, a_2].\left[\frac{1}{b_2}, \frac{1}{b_1}\right], 0 \notin [b_1, b_2]$

Example 3.5: Suppose $X = [a, b, c]$ and $Y = [p, q, r]$ be two fuzzy numbers whose membership functions are

$$\mu_x(X) = \begin{cases} \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \end{cases}$$

$$\mu_Y(X) = \begin{cases} \dfrac{x-p}{q-p} & p \le x \le q \\ \dfrac{r-x}{r-q} & q \le x \le r \end{cases}$$

Addition of Fuzzy Numbers: Then, $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $\alpha_Y = [(q-p)\alpha + p, r - (r-q)\alpha]$ are α-cuts of fuzzy members X and Y respectively. To calculate addition of fuzzy numbers X and Y, we first add the α-cuts of X and Y using interval arithmetic

$$\alpha_x + \alpha_y = [(b-a)\alpha + a, c - (c-b)\alpha] + [(q-p)\alpha + p, r - (r-q)\alpha]$$
$$= [a + p + (b - a + q - p)\alpha, c + r - (c - b + r - q)\alpha] \qquad (3.1)$$

To estimate the membership function $\mu_{X+Y(x)}$ it is need to equate to x both the first and second component in (3.1) as

$$x = [a + p + (b - a + q - p)\alpha] \,\&\, x = [c + r - (c - b + r - q)\alpha]$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.1) we get α together with the domain x.

$$\alpha = \frac{x - (a+p)}{(b+q) - (a+p)}, \qquad (a+p) \le x \le (b+q)$$
$$\alpha = \frac{(c+r) - x}{(c+r) - (b+q)}, \qquad (b+q) \le x \le (c+r)$$

Which gives

$$\mu_{X+Y}(x) = \begin{cases} \dfrac{x - (a+p)}{(b+q) - (a+p)} & (a+p) \le x \le (b+q) \\ \dfrac{(c+r) - x}{(c+r) - (b+q)} & (b+q) \le x \le (c+r) \end{cases}$$

Subtraction of Fuzzy Numbers: Let $X = [a, b, c]$ and $Y = [p, q, r]$ be two fuzzy numbers. $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $\alpha_Y = [(q-p)\alpha + p, r - (r-q)\alpha]$. To compute the subtraction of fuzzy numbers X and Y, it is needed to be first subtract the α-cuts of X and Y using interval arithmetic

$$\alpha_x - \alpha_y = [(b-a)\alpha + a, c - (c-b)\alpha] - [(q-p)\alpha + p, r - (r-q)\alpha]$$
$$= [(b-a)\alpha + a - (r - (r-q)\alpha), c - (c-b)\alpha - ((q-p)\alpha + p)]$$

$$= [(a-r)+(b-a+r-q)\alpha, (c-b)-(c-b+q-p)\alpha] \qquad (3.2)$$

To calculate the membership function $\mu_{X-Y(x)}$, it is needed to equate to x both the first and second component in (3.2) as

$$x = [(a-r)+(b-a+r-q)\alpha] \; \& \; x = [(c-b)-(c-b+q-p)\alpha]$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.2) we get α together with the domain x.

$$\alpha = \frac{x-(a-r)}{(b-q)-(a-r)}, \qquad (a-r) \le x \le (b-q)$$

$$\alpha = \frac{(c-p)-x}{(c-p)-(b-q)}, \qquad (b-q) \le x \le (c-p)$$

Which gives

$$\mu_{X-Y}(x) = \begin{cases} \dfrac{x-(a-r)}{(b-q)-(a-r)} & (a-r) \le x \le (b-q) \\ \dfrac{(c-p)-x}{(c-p)-(b-q)} & (b-q) \le x \le (c-p) \end{cases}$$

Multiplication of Fuzzy Numbers: Let $X = [a, b, c]$ and $Y = [p, q, r]$ be two fuzzy numbers. $\alpha_x = [(b-a)\alpha + a, c-(c-b)\alpha]$ and $\alpha_Y = [(q-p)\alpha + p, r-(r-q)\alpha]$. To calculate multiplication of fuzzy numbers X and Y, we first multiply the α-cuts of X and Y using interval arithmetic

$$\alpha_x * \alpha_y = [(b-a)\alpha + a, c-(c-b)\alpha] * [(q-p)\alpha + p, r-(r-q)\alpha]$$

$$= \big[\big((b-a)\alpha + a\big) * \big((q-p)\alpha + p\big), (c-(c-b)\alpha) * (r-(r-q)\alpha)\big] \qquad (3.3)$$

To find the membership function $\mu_{X*Y(x)}$ we equate to x both the first and second component in (3.3) which gives

$$x = (b-a)(q-p)\alpha^2 + \big((b-a)p + (q-p)a\big)\alpha + ap \; \&$$

$$\boldsymbol{x = (c-b)(r-q)\alpha^2 - \big((c-q)c + (c-b)r\big)\alpha + cr}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.3) we get α together with the domain x.

$$\alpha = \frac{-\big((b-a)p + (q-p)a\big) + \sqrt{\big((b-a)p + (q-p)a\big)^2 - 4(b-a)(q-p)ap}}{2(b-a)(q-p)}, ap \le x \le bq$$

and

$$\alpha = \frac{-\big((r-q)c + (c-b)r\big) - \sqrt{\big((r-q)c + (c-b)r\big)^2 - 4(c-b)(r-q)cr}}{2(c-b)(r-q)}, bq \le x \le cr$$

Which gives

$$\boldsymbol{\mu_{X*Y}(x) = \begin{cases} \dfrac{-\big((b-a)p + (q-p)a\big) + \sqrt{\big((b-a)p + (q-p)a\big)^2 - 4(b-a)(q-p)ap}}{2(b-a)(q-p)} & ap \le x \le bq \\ \dfrac{-\big((r-q)c + (c-b)r\big) - \sqrt{\big((r-q)c + (c-b)r\big)^2 - 4(c-b)(r-q)cr}}{2(c-b)(r-q)} & bq \le x \le cr \end{cases}}$$

Division of Fuzzy Numbers: Let $X = [a, b, c]$ and $Y = [p, q, r]$ be two positive fuzzy numbers. $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $\alpha_Y = [(q-p)\alpha + p, r - (r-q)\alpha]$. To calculate division of fuzzy numbers X and Y, we first divide the α-cuts of X and Y using interval arithmetic

$$\frac{\alpha_x}{\alpha_y} = \left[\frac{(b-a)\alpha + a, c - (c-b)\alpha}{(q-p)\alpha + p, r - (r-q)\alpha}\right] = [\frac{(b-a)\alpha + a}{r - (r-q)\alpha}, \frac{c - (c-b)\alpha}{(q-p)\alpha + p}] \quad (3.4)$$

To find the membership function $\mu_{X/Y(x)}$ we equate to x both the first and second component in (3.4) which gives

$$x = \frac{(b-a)\alpha + a}{r - (r-q)\alpha} \ \& \ x = \frac{c - (c-b)\alpha}{(q-p)\alpha + p}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (4.4), we get α together with the domain x.

$$\alpha = \frac{xr - a}{(b-a) + (q-r)x}, \frac{a}{r} \le x \le b/q \ \& \ \alpha = \frac{c - px}{(c-b) + (q-p)x}, \qquad b/q \le x \le c/p$$

Which gives

$$\mu_{X/Y}(x) = \begin{cases} \dfrac{xr - a}{(b-a) + (q-r)x}, & a/r \le x \le b/q \\ \dfrac{c - px}{(c-b) + (q-p)x}, & b/q \le x \le c/p \end{cases}$$

Inverse of Fuzzy Number: Let $X = [a, b, c]$ be a positive fuzzy number. Then, $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$. To calculate inverse of fuzzy number X, it should first take the inverse the α-cut of X using interval arithmetic

$$\frac{1}{\alpha_x} = \frac{1}{[(b-a)\alpha + a, c - (c-b)\alpha]} = \left[\frac{1}{c - (c-b)\alpha}, \frac{1}{(b-a)\alpha + a}\right] \quad (3.5)$$

To get the membership function $\mu_{1/X}$ we equate to x both the first and second component in (3.5), which gives $x = \frac{1}{c-(c-b)\alpha}$ & $x = \frac{1}{(b-a)\alpha+a}$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.5) we get α together with the domain x.

$$\alpha = \frac{cx - 1}{(c-b)x}, \qquad 1/c \le x \le 1/b \ \& \ \alpha = \frac{1 - ax}{(b-a)x}, \qquad 1/b \le x \le 1/a$$

Which gives

$$\mu_{1/X}(x) = \begin{cases} \dfrac{cx - 1}{(c-b)x}, & 1/c \le x \le 1/b \\ \dfrac{1 - ax}{(b-a)x}, & 1/b \le x \le 1/a \end{cases}$$

Exponential of a fuzzy number: Let $X = [a, b, c]$ be a positive fuzzy number. Then, $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$. To calculate exponential of the fuzzy number X, we first take the exponential the α-cut of X using interval arithmetic

$$\begin{aligned}\exp(\alpha_x) &= exp[(b-a)\alpha + a, c - (c-b)\alpha] \\ &= \left[\exp\left((b-a)\alpha + a\right), exp\left(c - (c-b)\alpha\right)\right] \qquad (3.6)\end{aligned}$$

To find the membership function $\mu_{\exp(x)}$ we equate to x both the first and second component in (4.6) which gives

$$x = \exp\big((b-a)\alpha + a\big) \ \& \ x = exp(c-(c-b)\alpha)$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.6) we get α together with the domain x.

$$\alpha = \frac{\ln(x) - a}{(b-a)}, \qquad \exp(a) \leq x \leq \exp(b) \ \& \ \alpha = \frac{c - \ln(x)}{c-b}, \qquad \exp(b) \leq x \leq \exp(c)$$

Which gives

$$\mu_{\exp(x)}(x) = \begin{cases} \dfrac{\ln(x) - a}{(b-a)}, & \exp(a) \leq x \leq \exp(b) \\ \dfrac{c - \ln(x)}{c-b}, & \exp(b) \leq x \leq \exp(c) \end{cases}$$

Logarithm of a fuzzy number:

Let $X = [a, b, c]$ be a positive fuzzy number. Then, $\alpha_x = [(b-a)\alpha + a, c-(c-b)\alpha]$. To calculate logarithm of the fuzzy number X, we first take the logarithm the α-cut of X using interval arithmetic

$$\ln(\alpha_x) = ln[(b-a)\alpha + a, c-(c-b)\alpha] = \big[\ln\big((b-a)\alpha + a\big), ln(c-(c-b)\alpha)\big] \qquad (3.7)$$

To find the membership function $\mu_{\ln(x)}$ we equate to x both the first and second component in (3.7) which gives

$$x = \ln\big((b-a)\alpha + a\big) \ \& \ x = ln(c-(c-b)\alpha)$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.7) we get α together with the domain x.

$$\alpha = \frac{\exp(x) - a}{(b-a)}, \ln(a) \leq x \leq \ln(b) \ \& \ \alpha = \frac{c - \exp(x)}{c-b}, \qquad \ln(b) \leq x \leq \ln(c)$$

Which gives

$$\mu_{\ln(x)}(x) = \begin{cases} \dfrac{\exp(x) - a}{(b-a)}, & \ln(a) \leq x \leq \ln(b) \\ \dfrac{c - \exp(x)}{c-b}, & \ln(b) \leq x \leq \ln(c) \end{cases}$$

Square root of a fuzzy number: Let $X = [a, b, c]$ be a positive fuzzy number. Then, $\alpha_x = [(b-a)\alpha + a, c-(c-b)\alpha]$. To calculate square root of the fuzzy number X, we first take the logarithm the α-cut of X using interval arithmetic

$$\sqrt{\alpha_x} = \sqrt{[(b-a)\alpha + a, c-(c-b)\alpha]} = \sqrt{(b-a)\alpha + a}, \sqrt{c-(c-b)\alpha} \qquad (3.8)$$

To find the membership function $\mu_{\sqrt{x}}(x)$ we equate to x both the first and second component in (3.8) which gives

$$x = \sqrt{(b-a)\alpha + a}, \ \& \ x = \sqrt{c-(c-b)\alpha}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (4.8) we get α together with the domain x.

$$\alpha = \frac{x^2 - a}{(b-a)}, \qquad \sqrt{a} \leq x \leq \sqrt{b} \ \& \ \alpha = \frac{c - x^2}{c-b}, \qquad \sqrt{b} \leq x \leq \sqrt{c}$$

Which gives

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$$\mu_{\sqrt{x}}(x) = \begin{cases} \dfrac{x^2 - a}{(b-a)}, & \sqrt{a} \le x \le \sqrt{b} \\ \dfrac{c - x^2}{c - b}, & \sqrt{b} \le x \le \sqrt{c} \end{cases}$$

nth root of a fuzzy number: Let $X = [a, b, c]$ be a positive fuzzy number. Then, $\alpha_x = [(b-a)\alpha + a, c - (c-b)\alpha]$. To calculate nth root of the fuzzy number X, we first take the nth root the α-cut of X using interval arithmetic

$$\sqrt[n]{\alpha_x} = \sqrt[n]{[(b-a)\alpha + a, c - (c-b)\alpha]} = \sqrt[n]{(b-a)\alpha + a} = \sqrt[n]{c - (c-b)\alpha} \qquad (3.9)$$

To find the membership function $\mu_{\sqrt{x}}(x)$ we equate to x both the first and second component in (3.9) which gives

$$x = \sqrt[n]{(b-a)\alpha + a} \ \ \& \ \ x = \sqrt[n]{c - (c-b)\alpha}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (4.9) we get α together with the domain x.

$$\alpha = \frac{x^n - a}{(b-a)}, \quad \sqrt[n]{a} \le x \le \sqrt[n]{b} \ \& \ \alpha = \frac{c - x^n}{c - b}, \quad \sqrt[n]{b} \le x \le \sqrt[n]{c}$$

Which gives

$$\mu_{\sqrt[n]{x}}(x) = \begin{cases} \dfrac{x^n - a}{(b-a)}, & \sqrt[n]{a} \le x \le \sqrt[n]{b} \\ \dfrac{c - x^n}{c - b}, & \sqrt[n]{b} \le x \le \sqrt[n]{c} \end{cases}$$

Example 3.6:

If $X = [1,2,4]$ and $Y = [3,5,6]$ be two fuzzy numbers whose membership functions are

$$\mu_X(x) = \begin{cases} x - 1 & 1 \le x \le 2 \\ \dfrac{4 - x}{2} & 2 \le x \le 4 \end{cases}$$

$$\mu_Y(x) = \begin{cases} \dfrac{x - 3}{2} & 3 \le x \le 5 \\ 6 - x & 5 \le x \le 6 \end{cases}$$

Then, $\alpha_X = [1 + \alpha, 4 - 2\alpha]$ and $\alpha_Y = [3 + 2\alpha, 6 - \alpha]$ are the $\alpha -$ cuts of fuzzy members X and Y respectively.

$$\boldsymbol{\alpha_X} + \boldsymbol{\alpha_Y} = [1 + \alpha, 4 - 2\alpha] + [3 + 2\alpha, 6 - \alpha] = [4 + 3\alpha, 10 - 3\alpha] \quad (3.10)$$

We take $x = 4 + 3\alpha$ And $x = 10 - 3\alpha$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.10) we get α together with the domain x.

$$\alpha = \frac{x - 4}{3}, \quad 4 \le x \le 7 \ \& \ \alpha = \frac{10 - x}{3}, \quad 7 \le x \le 10$$

$$\mu_{X+Y}(x) = \begin{cases} \dfrac{x - 4}{3}, & 4 \le x \le 7 \\ \dfrac{10 - x}{3}, & 7 \le x \le 10 \end{cases}$$

$$\boldsymbol{\alpha_X} - \boldsymbol{\alpha_Y} = [1 + \alpha, 4 - 2\alpha] - [3 + 2\alpha, 6 - \alpha] = [2\alpha - 5, 1 - 4\alpha] \quad (3.11)$$

We take $x = 2\alpha - 5$ And $x = 1 - 4\alpha$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.11) we get α together with the domain x.

$$\alpha = \frac{x+5}{2}, -5 \leq x \leq -3 \ \& \ \alpha = \frac{1-x}{4}, -3 \leq x \leq 1$$

$$\mu_{X-Y}(x) = \begin{cases} \frac{x+5}{2}, & -5 \leq x \leq -3 \\ \frac{1-x}{4}, & -3 \leq x \leq 1 \end{cases}$$

$$\boldsymbol{\alpha_X} * \boldsymbol{\alpha_Y} = [1+\alpha, 4-2\alpha] * [3+2\alpha, 6-\alpha] = [2\alpha^2 + 5\alpha + 3, 2\alpha^2 - 16\alpha + 24] \quad (3.12)$$

We take

$$x = 2\alpha^2 + 5\alpha + 3 \text{ And } x = 2\alpha^2 - 16\alpha + 24$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.12) we get α together with the domain x.

$$\alpha = \frac{-5 + \sqrt{25 - 4 * (2)(3-x)}}{2 * 2} = \frac{-5 + \sqrt{1+8x}}{4}, \quad 3 \leq x \leq 10$$

and

$$\alpha = \frac{16 - \sqrt{256 - 4 * (2)(24-x)}}{2 * 2} = \frac{8 - \sqrt{16+2x}}{2}, \quad 10 \leq x \leq 24$$

$$\mu_{X*Y}(x) = \begin{cases} \frac{-5+\sqrt{1+8x}}{4}, & 3 \leq x \leq 10 \\ \frac{8-\sqrt{16+2x}}{2}, & 10 \leq x \leq 24 \end{cases}$$

$$\frac{\boldsymbol{\alpha_x}}{\boldsymbol{\alpha_y}} = \left[\frac{1+\alpha, 4-2\alpha}{6-\alpha, 3+2\alpha}\right] = \left[\frac{1+\alpha}{6-\alpha}, \frac{4-2\alpha}{3+2\alpha}\right] \quad (3.13)$$

We take

$$x = \frac{1+\alpha}{6-\alpha} \text{ And } x = \frac{4-2\alpha}{3+2\alpha}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.13) we get α together with the domain x.

$$\alpha = \frac{6x-1}{1+x}, \frac{1}{6} \leq x \leq \frac{2}{5} \ \& \ \alpha = \frac{4-3x}{2(1+x)}, \frac{2}{5} \leq x \leq \frac{4}{3}$$

$$\mu_{X/Y}(x) = \begin{cases} \frac{6x-1}{1+x}, & \frac{1}{6} \leq x \leq \frac{2}{5} \\ \frac{4-3x}{2(1+x)}, & \frac{2}{5} \leq x \leq \frac{4}{3} \end{cases}$$

$$\frac{\mathbf{1}}{\boldsymbol{\alpha_x}} = \frac{1}{[1+\alpha, 4-2\alpha]} = \left[\frac{1}{4-2\alpha}, \frac{1}{1+\alpha}\right] \quad (3.14)$$

We take

$$x = \frac{1}{4-2\alpha} \text{ and } x = \frac{1}{1+\alpha}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.14) we get α together with the domain x.

$$\alpha = \frac{4x-1}{2x}, \quad \frac{1}{4} \le x \le \frac{1}{2} \ \& \ \alpha = \frac{1-x}{x}, \quad \frac{1}{2} \le x \le 1$$

$$\mu_{1/X}(x) = \begin{cases} \frac{4x-1}{2x}, & \frac{1}{4} \le x \le \frac{1}{2} \\ \frac{1-x}{x}, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\boldsymbol{exp(\alpha_X)} = \exp([1+\alpha, 4-2\alpha]) = [\exp(1+\alpha), \exp(4-2\alpha)] \qquad (3.15)$$

We take

$$x = \exp(1+\alpha) \text{ and } x = \exp(4-2\alpha)$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (4.15) we get α together with the domain x.

$$\alpha = lnx - 1, \quad \exp(1) \le x \le \exp(2) \ \& \ \alpha = \frac{4 - lnx}{2}, \quad \exp(2) \le x \le \exp(4)$$

$$\mu_{\exp(X)}(x) = \begin{cases} lnx - 1, & \exp(1) \le x \le \exp(2) \\ \frac{4 - lnx}{2}, & \exp(2) \le x \le \exp(4) \end{cases}$$

$$\boldsymbol{\alpha_x} = ln[1+\alpha, 4-2\alpha] = [\ln(1+\alpha), \ln(4-2\alpha)] \qquad (3.16)$$

We take $x = \ln(1+\alpha)$ And $x = \ln(4-2\alpha)$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.16) we get α together with the domain x.

$$\alpha = \exp(x) - 1, \quad ln1 \le x \le ln2 \ \& \ \alpha = \frac{4 - \exp(x)}{2}, \quad ln2 \le x \le ln4$$

$$\mu_{lnX}(x) = \begin{cases} \exp(x) - 1, & ln1 \le x \le ln2 \\ \frac{4 - \exp(x)}{2}, & ln2 \le x \le ln4 \end{cases}$$

$$\sqrt{\boldsymbol{\alpha_x}} = [\sqrt{1+\alpha, 4-2\alpha}] = \left[\sqrt{(1+\alpha)}, \sqrt{(4-2\alpha)}\right] \qquad (3.17)$$

We take

$$x = \sqrt{(1+\alpha)} \text{ And } x = \sqrt{(4-2\alpha)}$$

Now, expressing α in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3.17) we get α together with the domain x.

$$\alpha = x^2 - 1, \quad 1 \le x \le \sqrt{2} \ \& \ \alpha = \frac{4 - x^2}{2}, \quad \sqrt{2} \le x \le 2$$

$$\mu_{\sqrt{X}}(x) = \begin{cases} x^2 - 1, & 1 \le x \le \sqrt{2} \\ \frac{4 - x^2}{2}, & \sqrt{2} \le x \le 2 \end{cases}$$

While

$$\mu_{\sqrt[n]{X}}(x) = \begin{cases} x^n - 1, & 1 \le x \le \sqrt[n]{2} \\ \frac{4 - x^n}{2}, & \sqrt[n]{2} \le x \le \sqrt[n]{4} \end{cases}$$

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Multiplication of Fuzzy set by a crisp number: Assume $A(x)$ be a fuzzy set and λ is a crisp number, then $\lambda . A(x) = \left\{\left(x, \lambda . A(x)\right)\right), x \in X\}$

4 Fuzzy Numbers in the set of integers:

When the membership function $\alpha = \mu_A(x)$ has argument $x \in Z$, where Z is the set of integers the projections of the α-level intervals ($\alpha - cuts$), $A_\alpha \in [a_1^{(\alpha)}, a_2^{(\alpha)}]$ belong to Z.

To represent a fuzzy member A with maximum in Z as

$$A = [a_1, a_2] = \{a_1, a_1^{(\alpha_1)}, a_1^{(\alpha_2)}, \dots., a_1^{(\alpha_n)}, a_M, a_2^{(\alpha_n)}, \dots, a_2^{(\alpha_2)}, a_2^{(\alpha_1)}, a_2\}$$

The membership function $\alpha = \mu_A(x)$

x	a_1	$a_1^{(\alpha_1)}$	...	$a_1^{(\alpha_n)}$	a_M	$a_2^{(\alpha_n)}$	...	$a_2^{(\alpha_1)}$	a_2
α	0	α_1	...	α_n	1	α_n	...	α_1	0

Where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. The point $(a_M, 1)$ is the maximum of A.

The α_i $-$level consists of the endpoints $(a_1^{(\alpha_i)}, \alpha_i)$, $(a_2^{(\alpha_i)}, \alpha_i)$ and all points of levels higher than α_i projected on level α_i; we denote it by

$$A_\alpha = \left\{a_1^{(\alpha_i)}, a_1^{(\alpha_{i+1})}, \dots, a_1^{(\alpha_n)}, a_M, a_2^{(\alpha_{i+1})}, a_2^{(\alpha_i)}\right\}_{\alpha_i}$$

Example: 4.1 Consider the incomplete fuzzy number A in Z given by the table

A	x	0	1	2	3	4	5	6	7
	α	0	0.1	0.3	0.8	1	0.7	0.3	0

Graphically shown that, at the levels $\alpha = 0.1,0.7,0,8$ only one endpoint is given. Applying the procedure of redefining above and incorporating in addition the level $\alpha = 0.5$, obtain the extended table

A	x	0	1	2	3	3	3	4	4	5	5	6	6	7
	α	0	0.1	0.3	0.5	0.7	0.8	1	0.8	0.7	0.5	0.3	0.1	0

The supplemented points are marked by circle in above circles. For instance, at level levels $\alpha = 0.1$ the right endpoint is missing. We use instead the right endpoint 6 of next higher level $\alpha = 0.3$. Similarly, we find the point 3 at level levels $\alpha = 0.7$ and the point 4 at level levels $\alpha = 0.8$. level levels $\alpha = 0.5$ is not presented initially. We use the endpoints at the higher level levels $\alpha = 0.7$..i.e 3 and 5.

The fuzzy number A is a discrete function while the complete fuzzy number is a relation since for each $x = 3,4,5,6$ there corresponds more than one value for levels α

Example 4.2: Consider the table

x \ y	-3	-2	-1	0	1	2
-1	0	0	0	0	0	0
0	0	0.2	0.4	0.4	0.2	0
1	0	0.4	0.8	1	0.6	0
2	0	0.2	0.6	0.8	0.6	0
3	0	0	0	0	0	0

The table represents an incomplete fuzzy number A in Z^2. To complete a level, we have to project the points at higher level. The $\alpha -$ level domain or α $-$cuts of the complete fuzzy number in Z^2 are presented below.

$x \backslash y$	-3	-2	-1	0	1	2
-1	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0

0-cut

$x \backslash y$	-2	-1	0	1
0	0.2	0.2	0.2	0.2
1	0.2	0.2	0.2	0.2
2	0.2	0.2	0.2	0.2

0.2-cut

$x \backslash y$	-2	-1	0	1
0		0.4	0.4	0.4
1	0.4	0.4	0.4	0.4
2		0.4	0.4	0.4

0.4-cut

$x \backslash y$	-1	0	1
1	0.6	0.6	0.6
2	0.6	0.6	0.6

0.6-cut

$x \backslash y$	-1	0
1	0.8	0.8
2	0.8	

0.8-cut

$x \backslash y$	0
1	1

1-cut

Example 4.3: Consider the discrete fuzzy numbers A and B

A	x	-2	-1	0	1	2
	α	0	0.3	0.8	1	0.3
B	x	0	1	3	4	5
	α	0.3	0.7	1	0.2	0

Represent these tables, it is noticed that, they are incomplete and incompatible. To make them compatible by using the redefine procedure. We construct the completed tables.

A	x	-2	-1	0	0	1	2	2
	α	0	0.3	0.8	1	1	0.3	0

B	x	0	0	0	1	3	3	4	5
	α	0	0.2	1	0.7	0.7	1	0.2	0

Example 4.4: Consider the discrete fuzzy numbers A and B in Z

$A =$	x	-2	0	1	2	4	6
	α	0	0.2	0.5	1	0.7	0.2

$B =$	x	0	2	3	4	5	6	7	8
	α	0.3	0.5	0.8	1	0.5	0.3	0.2	0

Then, for $\alpha = 0.2, 0.3, 0.7$ *and* 0.8, find $[A+B]_\alpha$ and $[A-B]_\alpha$

	x	-2	0	1	1	2	2	2	2	4	4	4	6	6

$\underset{\approx}{A}$	α	0	0.2	0.3	0.5	0.7	0.8	1	0.8	0.7	0.5	0.3	0.2	0

$\underset{\approx}{B}$	x	0	0	0	2	3	3	4	4	4	5	6	7	8
	α	0	0.2	0.3	0.5	0.7	0.8	1	0.8	0.7	0.5	0.3	0.2	0

$[A+B]_\alpha = A_\alpha + B_\alpha$

If $\alpha = 0.2$: $[A+B]_{0.2} = A_{0.2} + B_{0.2} = [0,6]_{0.2} + [0,7]_{0.2} = [0,13]_{0.2}$

If $\alpha = 0.3$: $[A+B]_{0.3} = A_{0.3} + B_{0.3} = [1,4]_{0.3} + [0,6]_{0.3} = [1,10]_{0.3}$

If $\alpha = 0.7$: $[A+B]_{0.7} = A_{0.7} + B_{0.7} = [2,4]_{0.7} + [3,4]_{0.7} = [5,8]_{0.7}$

If $\alpha = 0.8$: $[A+B]_{0.8} = A_{0.8} + B_{0.8} = [2,2]_{0.8} + [3,4]_{0.7} = [5,6]_{0.8}$

$[A-B]_\alpha = A_\alpha - B_\alpha$

If $\alpha = 0.2$:$[A-B]_{0.2} = A_{0.2} - B_{0.2} = [0,6]_{0.2} - [0,7]_{0.2} = [-7,6]_{0.2}$

If $\alpha = 0.3$: $[A-B]_{0.3} = A_{0.3} - B_{0.3} = [1,4]_{0.3} - [0,6]_{0.3} = [-5,4]_{0.3}$

If $\alpha = 0.7$:$[A-B]_{0.7} = A_{0.7} - B_{0.7} = [2,4]_{0.7} - [3,4]_{0.7} = [-2,1]_{0.7}$

If $\alpha = 0.8$: $[A-B]_{0.8} = A_{0.8} - B_{0.8} = [2,2]_{0.8} - [3,4]_{0.8} = [-2,-1]_{0.8}$

5 Fuzzy Relations: Consider the cartesian product $A \times B = \{(x,y)|\ x \in A, y \in B\}$ Where $A \subseteq U_1$ and $B \subseteq U_2$, the fuzzy relation on $A \times B$ is denoted by $\mathcal{R}$, or define as the set

$$\mathcal{R} = \{\big((x,y), \mu_{\mathcal{R}}(x,y)\big)|(x,y) \in A \times B, \mu_{\mathcal{R}}(x,y) \in [0,1]\}$$

Example 5.1: If

$$R = \{\big((x_1,y_1),0.1\big),\big((x_1,y_2),0.2\big),\big((x_1,y_3),0.1\big),\big((x_2,y_1),0.3\big),\big((x_2,y_2),0.5\big),\big((x_2,y_3),0.4\big)\}$$

Can be written as

$x \backslash y$	y_1	y_2	y_3
x_1	0.1	0.2	0.1
x_2	0.3	0.5	0.4

Remark

Identity Relation I is defined for all $(x,y) \in A \times B$ by membership function as follows

$$I = \mu_I = \begin{cases} 1 & for\ x = 1 \\ 0 & otherwise \end{cases}$$

5.1 Basic Operation on Fuzzy Relation: Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations on $A \times B$ such that:

$$\mathcal{R}_1 = \{\Big((x,y), \mu_{\mathcal{R}_1}(x,y)\Big)|(x,y) \in A \times B, \mu_{\mathcal{R}_1}(x,y) \in [0,1]\}$$

$$\mathcal{R}_2 = \{\Big((x,y), \mu_{\mathcal{R}_2}(x,y)\Big)|(x,y) \in A \times B, \mu_{\mathcal{R}_2}(x,y) \in [0,1]\}$$

5.2 Equality:$\mathcal{R}_1 = \mathcal{R}_2$ if and only if for every pairs $(x,y) \in A \times B$, $\mu_{\mathcal{R}_1}(x,y) = \mu_{\mathcal{R}_2}(x,y)$

5.3 Inclusion: If for every pairs $(x,y) \in A \times B$, $\mu_{\mathcal{R}_1}(x,y) \le \mu_{\mathcal{R}_2}(x,y)$, then the relation $\mathcal{R}_1$ is included in $\mathcal{R}_2$ $\mathcal{R}_1 \subseteq \mathcal{R}_2$ if at least one pair (x,y) such that $\mu_{\mathcal{R}_1}(x,y) < \mu_{\mathcal{R}_2}(x,y)$

Example 5.2: For the relations, if $\mathcal{R}_1$and $\mathcal{R}_2$ are presented in the following tables respectively:

$x \backslash y$	y_1	y_2	y_3
x_1	0	0.2	0.6
x_2	0.4	1	0.8

$x \backslash y$	y_1	y_2	y_3
x_1	0.1	0.2	0.7
x_2	0.5	1	0.9

$\mu_{\mathcal{R}_1}(x_1,y_1) = 0 < \mu_{\mathcal{R}_2}(x_1,y_1) = 0.1, \mu_{\mathcal{R}_1}(x_1,y_2) = 0.2 = \mu_{\mathcal{R}_2}(x_1,y_2) = 0.2$
$\mu_{\mathcal{R}_1}(x_1,y_3) = 0.6 < \mu_{\mathcal{R}_2}(x_1,y_3) = 0.7, \mu_{\mathcal{R}_1}(x_2,y_1) = 0.4 < \mu_{\mathcal{R}_1}(x_2,y_1) = 0.5$
$\mu_{\mathcal{R}_1}(x_2,y_2) = 1.0 = \mu_{\mathcal{R}_2}(x_2,y_2) = 1.0, \mu_{\mathcal{R}_1}(x_2,y_3) = 0.8 < \mu_{\mathcal{R}_2}(x_2,y_3) = 0.9$. Therefore, $\mathcal{R}_1 \subset \mathcal{R}_2$

Complementation: The complement of a relation $\mathcal{R}$, denoted by $\bar{\mathcal{R}}$ is defined by $\mu_{\bar{\mathcal{R}}}(x,y) = 1 - \mu_{\mathcal{R}}(x,y)$

The Intersection: The intersection of a relation $\mathcal{R}_1$ and $\mathcal{R}_2$, is defined by

$$\mu_{\mathcal{R}_1 \cap \mathcal{R}_2}(x,y) = \min\{\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(x,y), (x,y) \in A \times B\}$$

The Union: The union of a relation $\mathcal{R}_1$ and $\mathcal{R}_2$, is defined by

$$\mu_{\mathcal{R}_1 \cup \mathcal{R}_2}(x,y) = \max\{\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(x,y), (x,y) \in A \times B\}$$

Example 5.3: Consider relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by the following tables respectively

$x \backslash y$	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0	0.7	0.3
x_3	0.2	0.8	1

$x \backslash y$	y_1	y_2	y_3
x_1	0.3	0.3	0.2
x_2	0.5	0	1
x_3	0.7	0.3	0.1

$\bar{\mathcal{R}}_1 =$

$x \backslash y$	y_1	y_2	y_3
x_1	1	0.9	0.8
x_2	1	0.3	0.7
x_3	0.8	0.2	0

$\bar{\mathcal{R}}_2 =$

$x \backslash y$	y_1	y_2	y_3
x_1	0.7	0.7	0.8
x_2	0.5	1	0
x_3	0.3	0.7	0.9

$\mu_{\mathcal{R}_1 \cap \mathcal{R}_2}(x,y) =$

$x \backslash y$	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0	0	0.3
x_3	0.2	0.3	0.1

$\mu_{\mathcal{R}_1 \cup \mathcal{R}_2}(x,y) =$

$x \backslash y$	y_1	y_2	y_3
x_1	0.3	0.3	0.2
x_2	0.5	0.7	1
x_3	0.7	0.8	1

5.4 Direct Product:

If the fuzzy relations are reduced as fuzzy sets as:

$$A(x) = \{(x, \mu_A(x)), \mu_A(x) \in [0,1], x \in A \subset U_1\}$$

&

$$B(x) = \{(x, \mu_B(x)), \mu_B(x) \in [0,1], x \in B \subset U_1\}$$

The cartesian product $A \times B = \{(x, y), \min(\mu_A(x), \mu_B(x)), (x, y) \in A \times B\}$

Whereas, the direct product $A \times B = \{(x, y), \max(\mu_A(x), \mu_B(x)), (x, y) \in A \times B\}$

Example 5.4: Given the fuzzy sets

$$\mathcal{R}_1 = A(x) = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.1)\} \,\&\, \mathcal{R}_2 = B(x) = \{(y_1, 0.3), (y_2, 0.5)\}$$

To find direct min Product and direct max direct as

$Min\ product: A \times B =$

$x \backslash y$	y_1	y_2
x_1	0.3	0.4
x_2	0.3	0.5
x_3	0.1	0.1

Max product: $A \times B =$

$x \backslash y$	y_1	y_2
x_1	0.3	0.4
x_2	0.3	0.5
x_3	0.1	0.1

5.5 Projections of a fuzzy Relation:

Assume the fuzzy relation $\mathcal{R} = \{(x, y), \mu_{\mathcal{R}}(x, y)\}$, if the first projection of $\mathcal{R}$ is defined as

$$\mathcal{R}^{(1)} = \{(x, y), \mu_{\mathcal{R}^{(1)}}(x, y) = \{(x, \max_y \mu_{\mathcal{R}}(x, y))|(x, y) \in A \times B\}$$

The second projection of $\mathcal{R}$ is defined as

$$\mathcal{R}^{(2)} = \{(x, y), \mu_{\mathcal{R}^{(2)}}(x, y) = \{(y, \max_x \mu_{\mathcal{R}}(x, y))|(x, y) \in A \times B\}$$

The total projection is defined as $\mathcal{R}^{(T)} = \max_x \max_y \{\mu_{\mathcal{R}}(x, y)|(x, y) \in A \times B\}$.

Example 5.5: For the following relation $\mathcal{R}$ find the first, second and total projections if,

$\mathcal{R}$	y_1	y_2	y_3	y_4	y_5	y_6	$\mathcal{R}^{(1)}$
x_1	0	0.1	0.3	0.7	0.9	1	1
x_2	0.8	0.6	0.4	0.2	0	0.2	0.8
$\mathcal{R}^{(2)}$	0..8	0.6	0.4	0.7	0.9	1	1=$\mathcal{R}^{(T)}$

$\mathcal{R}^{(1)} = \{1, 0.8) \,\&\, \mathcal{R}^{(2)} = \{0.8, 0.6, 0.4, 0.7, 0.9, 1\}, \mathcal{R}^{(T)} = 1$

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Among all those relations having a common projection is called cylindrical extension of the projection. In the previous example the cylindrical extensions of both $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are presented in both tables below:

$C(\mathcal{R}_1)$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1	1	1	1	1	1
x_2	0.8	0.8	0.8	0.8	0.8	0.8

$C(\mathcal{R}_2)$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0..8	0.6	0.4	0.7	0.9	1
x_2	0..8	0.6	0.4	0.7	0.9	1

5.6 Max-Min and Min-Max Compositions

The operations composition combines fuzzy relations in different variables if (x,y) & (y,z); $x \in A, y \in B, z \in C$. Consider the relations

$$\mathcal{R}_1(x,y) = \{(x,y), \mu_{\mathcal{R}_1}(x,y) | (x,y) \in A \times B\} \,\&\, \mathcal{R}_2(y,z) = \{(y,z), \mu_{\mathcal{R}_2}(y,z) | (y,z) \in B \times C\}$$

The Max-Min composition denoted $\mathcal{R}_1 \circ \mathcal{R}_2$ with membership function $\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}$is defined by

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x,z), \max\left(\min\left(\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(y,z)\right)\right), (x,z) \in A \times C, y \in B\}$$

Hence it is a relation in the domain $A \times C$

Example 5.6: The Fuzzy Relation s are given by

$\mathcal{R}_1$	y_1	y_2
x_1	0.6	0.5
x_2	0.1	1
x_3	0	0.7

$\mathcal{R}_2$	z_1	z_2	z_2
y_1	0.7	0.3	0.4
y_2	0.9	0.1	0.6

$$\min\left(\mu_{\mathcal{R}_1}(x_1,y_1), \mu_{\mathcal{R}_2}(y_1,z_1)\right) = \min(0.6,0.7) = 0.6$$

$$\min\left(\mu_{\mathcal{R}_1}(x_1,y_2), \mu_{\mathcal{R}_2}(y_2,z_1)\right) = \min(0.5,0.9) = 0.5$$

$$Max(0.6,0.5) = 0.6$$

$$\min\left(\mu_{\mathcal{R}_1}(x_2,y_1), \mu_{\mathcal{R}_2}(y_1,z_1)\right) = \min(0.1,0.7) = 0.1$$

$$\min\left(\mu_{\mathcal{R}_1}(x_2,y_2), \mu_{\mathcal{R}_2}(y_2,z_1)\right) = \min(1,0.9) = 0.9$$

$$Max(0.1,0.9) = 0.9$$

$$\min\left(\mu_{\mathcal{R}_1}(x_3,y_1), \mu_{\mathcal{R}_2}(y_1,z_1)\right) = \min(0,0.7) = 0$$

$$\min\left(\mu_{\mathcal{R}_1}(x_3,y_2), \mu_{\mathcal{R}_2}(y_2,z_1)\right) = \min(0.7,0.9) = 0.7$$

$$Max(0,0.7) = 0.7$$

The rest we follow same procedure, we obtain

$\mathcal{R}_1 \circ \mathcal{R}_2$	z_1	z_2	z_3
x_1	0.6	0.3	0.5
x_2	0.9	0.1	0.6

x_3 | 0.7 0.1 0.6

Example 5.7: If fuzzy sets

$X = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.5)\}$ and $Y = \{(y_1, 0.2), (y_2, 0.8), (y_3, 0.4)\}$, then

1. Find Cartesian product of these fuzzy sets

2. Write the Cartesian product of X and Y in the matrix form. Find the first projection (over X), second projection (over Y), cylinder extensions according to first projection and second projection

1) Then, $X \times Y = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.5)\} \times \{(y_1, 0.2), (y_2, 0.8), (y_3, 0.4)\}$

$$= \left\{ \begin{matrix} ((x_1, y_1), 0.2), ((x_1, y_2), 0.4), ((x_1, y_3), 0.4), \\ ((x_2, y_1), 0.2), ((x_2, y_2), 0.6), ((x_2, y_3), 0.4), \\ ((x_3, y_1), 0.2), ((x_3, y_2), 0.5), ((x_3, y_3), 0.4) \end{matrix} \right\}$$

2)

$X \times Y$	y_1	y_1	y_3
x_1	0.2	0.4	0.4
x_2	0.2	0.6	0.4
x_3	0.2	0.5	0.4

First projection $R^{(1)} = (0.4,\ 0.6,\ \ 0.5)^T$, $R^{(2)} = (0.2,\ 0.6\ \ 0.4)$, $R(T) = 0.6$

The second projection

$C(R^{(1)})$	y_1	y_1	y_3
x_1	0.4	0.4	0.4
x_2	0.6	0.6	0.6
x_3	0.5	0.5	0.5

$C(R^{(2)})$	y_1	y_1	y_3
x_1	0.2	0.6	0.4
x_2	0.2	0.6	0.4
x_3	0.2	0.6	0.4

Therefore, $RCC(R^{(1)})$ and $RCC(R^{(2)})$

Example 5.8:

Suppose that the universe X, Y, Z and T are given as $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ $Z = \{z_1, z_2, z_3\}$ and $T = \{t_1, t_2, t_3\}$. Let us consider three fuzzy relations R_1 on $X \times Y$ and R_2 on $Y \times Z$ and R_3 on $Z \times T$ such as:

$$R_1 = \begin{pmatrix} 1 & 0.4 & 0.6 \\ 0.5 & 0.8 & 0.7 \\ 0.9 & 0.6 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0.3 & 0.7 & 0.9 \\ 0.6 & 1 & 0.4 \\ 0.2 & 0.9 & 1 \end{pmatrix} \text{ and } R_3 = \begin{pmatrix} 0.4 & 0.3 & 0.5 \\ 0.8 & 0.2 & 0.7 \\ 0.6 & 0.3 & 1 \end{pmatrix}.$$

Then, $\overline{R_1 \circ R_2} \bullet \mathrm{R}_3 =?$ ($\circ$ denotes max-min operation, $\bullet$ denotes min-max operation)?

$$\overline{R_1 \circ R_2} =, \overline{R_1} \boxdot \overline{R_2} = \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0 \end{pmatrix} \boxdot \begin{pmatrix} 0.7 & 0.3 & 0.1 \\ 0.4 & 0 & 0.6 \\ 0.8 & 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0 \end{pmatrix}$$

$$\overline{R_1 \circ R_2} \bullet \mathrm{R}_3 = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 0.4 & 0.3 & 0.5 \\ 0.8 & 0.2 & 0.7 \\ 0.6 & 0.3 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.5 \\ 0.4 & 0.2 & 0.5 \end{pmatrix}$$

Example 5.9: Consider the discrete fuzzy numbers A and B in Z.

$A =$	x	-1	0	1	3	4	5	6	7
	α	0	0.4	0.5	0.8	1	0.6	0.5	0.2
$B =$	x	1	2	3	4	5	6	7	8
	α	0.2	0.3	0.7	0.9	1	0.8	0.2	0

Then, find $[A + B]_{0.6}$, $[A - B]_{0.3}$ ve $[AB]_{0.7}$

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A	x	-1	0	0	0	1	3	3	3	4	4	4	4	5	6	6	7	7
$\approx$	α	0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0

B	x	1	1	2	3	3	3	3	4	4	5	5	6	6	6	6	6	6	7	7
$\approx$	α	0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.8	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0

$$[A+B]_{0.6} = A_{0.6} + B_{0.6} = [3,5]_{0.6} + [3,6]_{0.6} = [6,11]_{0.6}$$

$$[A-B]_{0.3} = A_{0.3} - B_{0.3} = [0,6]_{0.3} + [-6,-2]_{0.3} = [-6,4]_{0.3}$$

$$[AB]_{0.7} = A_{0.7}B_{0.7} = [3,4][3,6] = \{\min[9,18,12,24], \max[9,18,12,24]\} = [9,24]_{0.7}$$

6 Conclusion

This paper focuses on three cases on fuzzy sets, operations on fuzzy sets, fuzzy numbers and fuzzy relations. The results of the study warrant researchers how to figure out in fuzzy set-in n- dimensional and think about fuzzy graphs in generalized. The results in this study are not only important to using relations in fuzzy sets and numbers in creating adaptive intelligent learning, but deriving us also to look at cluster fuzzy sets in our plan as future work.

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